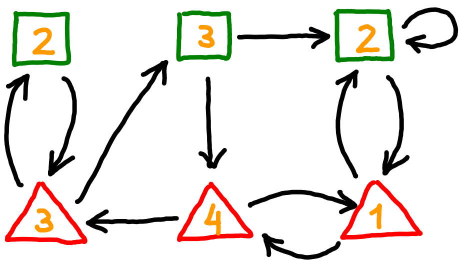
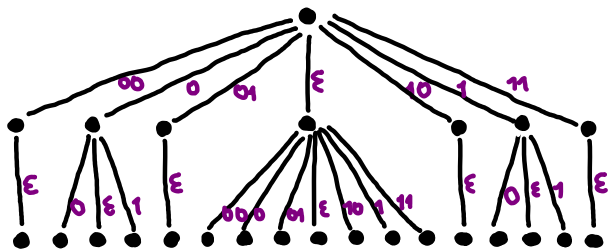
UNIVERSAL TREES

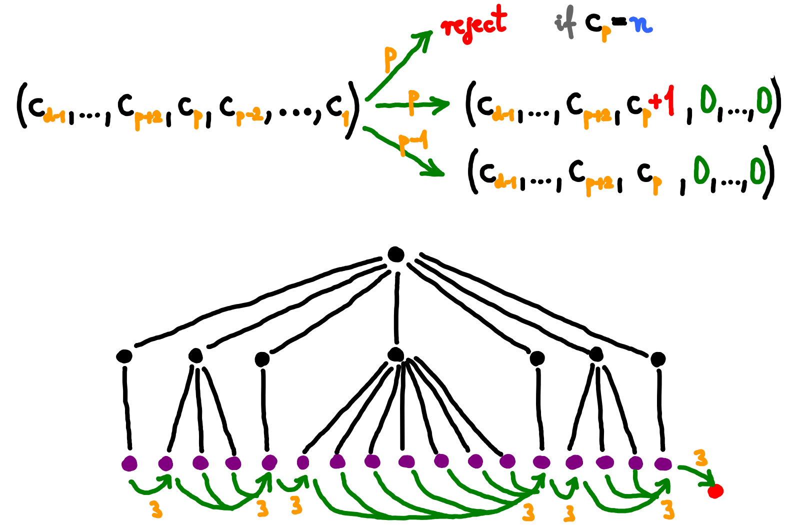
QUASI-POLYNOMIAL

PARITY GAMES

SEPARATING AUTOMATA







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UNIVERSITY OF WARWICK

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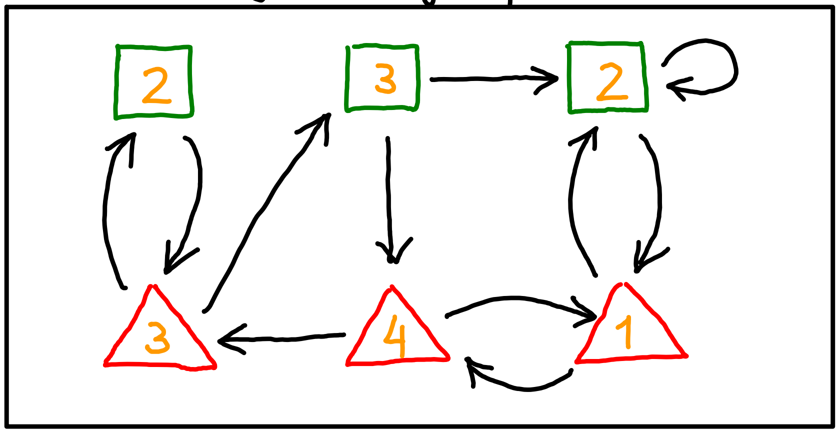
K. LEHTINEN, R. MORVAN, THEJASWINI K.S.

PARITY GAMES

$n = |V|$

A game graph



$$G = (V = V_{Even} \uplus V_{Odd}, E, \pi)$$

$$\pi : V \to \{1, 2, 3, 4, 5, \dots, d\}$$

PARITY GAMES

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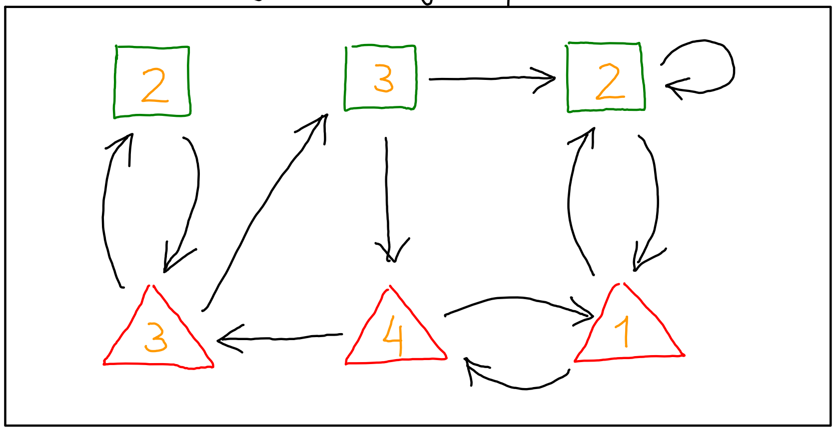

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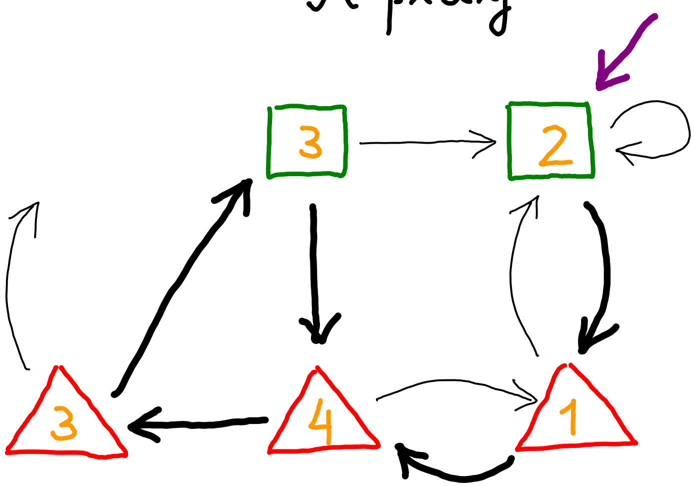
A play



Even wins $\langle p_1, p_2, p_3, \dots \rangle$

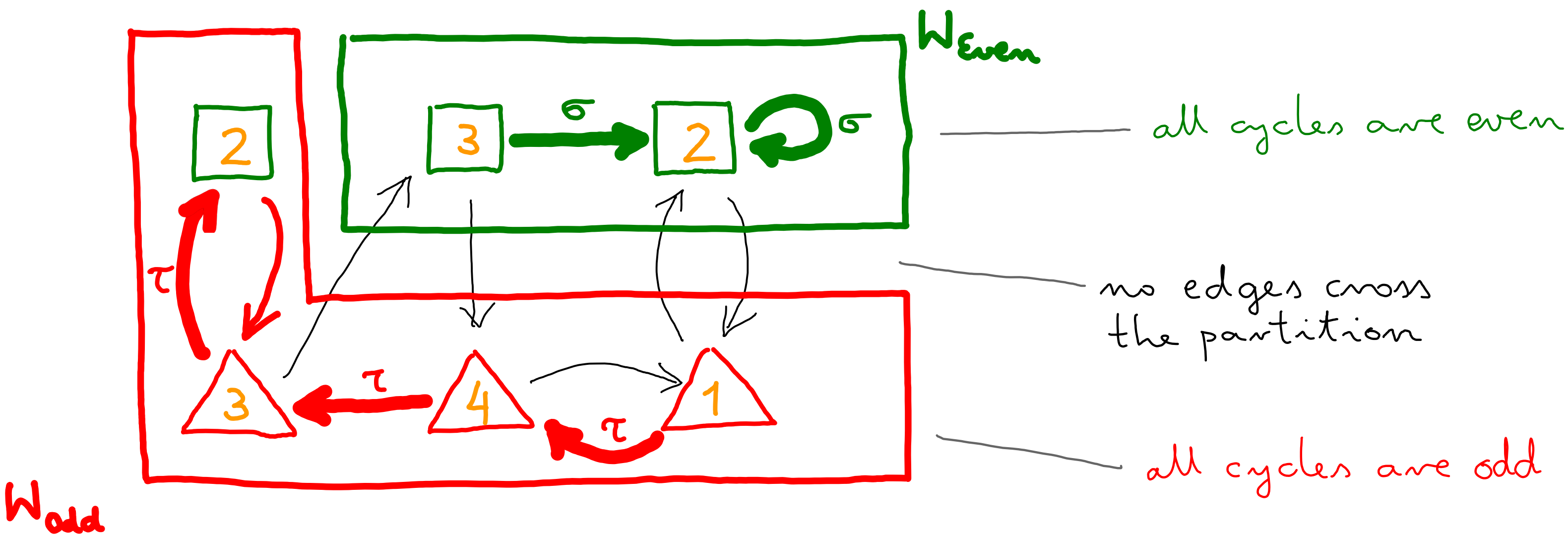
iff

$\left[\limsup_{i \to \infty} p_i\right]$ is even

SHORT WITNESSES FROM POSITIONAL DETERMINACY

THEOREM [Emerson, Jutla 1991; Mostowski 1991; re-proved since 1960's]

Parity games are positionally determined



COROLLARY [Emerson, Jutla, Sistla 1993]

Deciding the winner in parity games is $\boxed{\text{in NP} \cap \text{co-NP}}$

ALGORITHMS FOR SOLVING PARITY GAMES

- $\boxed{n^{d+O(1)}}$ [McNaughton 1993; Zielonka 1998]
- $\boxed{n^{\frac{d}{2}+O(1)}}$ [Browne, Clarke, Jha, Long, Marrero 1994; Seidl 1996; J. 2000]
- $\boxed{n^{d+O(1)}}$ strategy iteration [Vöge, J. 2000]

  $\boxed{2^{\Omega(n)}}$ [Friedmann 2009]

  - policy iteration for MDPs [Fearnley 2010]
  - randomized simplex [Hansen, Friedmann, Zwick 2011]
- $\boxed{n^{O(\sqrt{n})}}$ [Björklund, Sandberg, Vorobyov 2003; J., Paterson, Zwick 2006]
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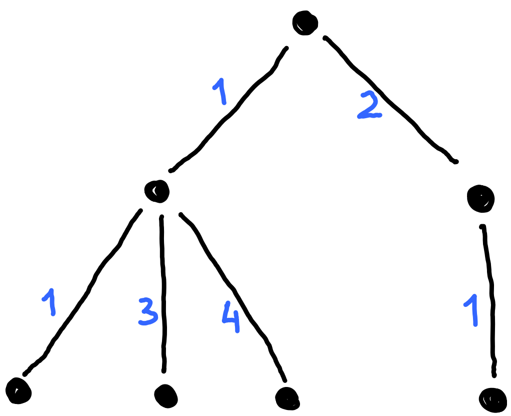
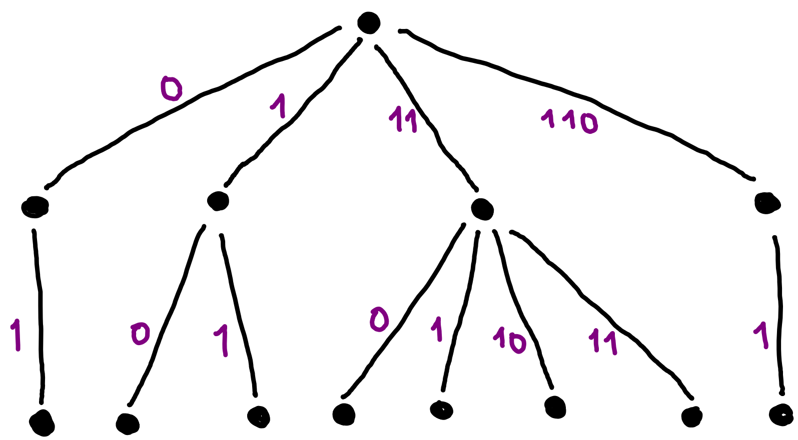
- $n^{\lg d+O(1)}$ **[Calude, Jain, Khoussainov, Li, Stephan 2017; J., Lazić 2017; Lehtinen 2018]**

ORDERED TREES

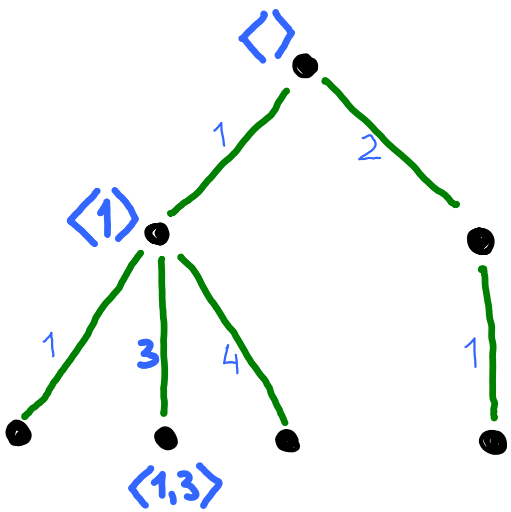


$(\mathbb{N}, <)$

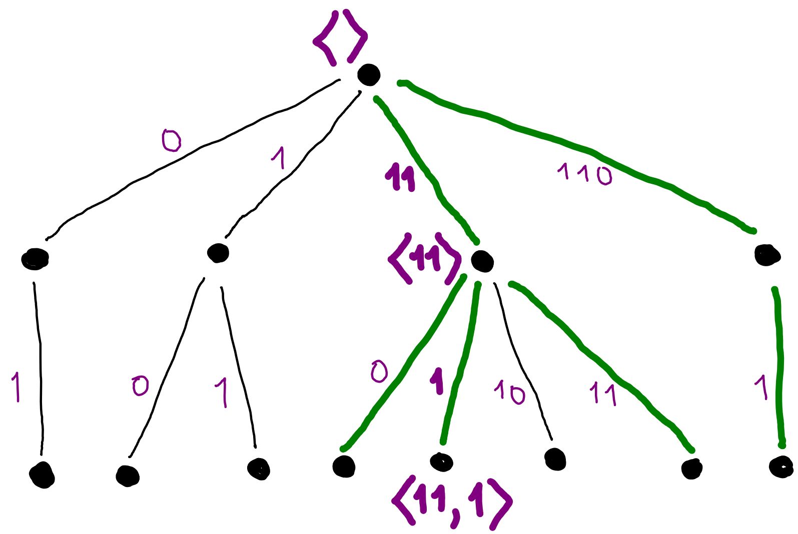


$(\mathbb{B}^*, <_{lex})$

ORDERED TREES



embeds into



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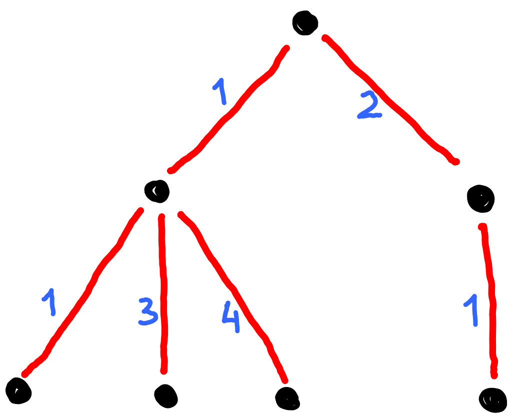
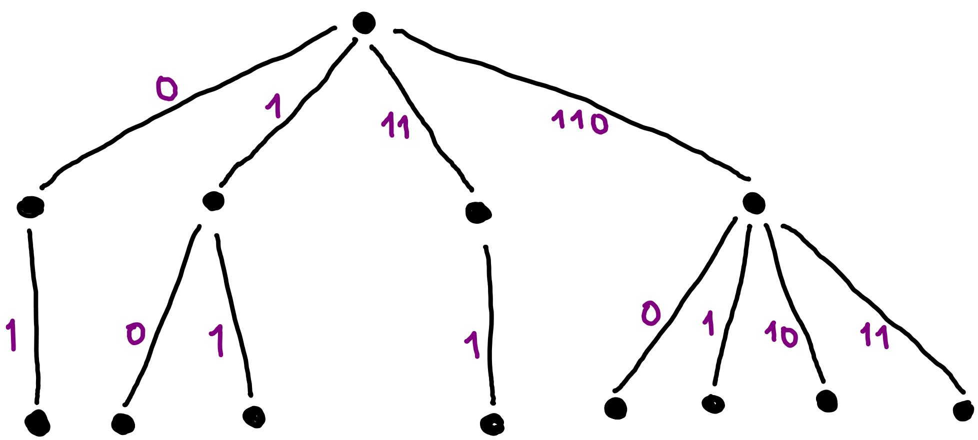
ORDERED TREES



does not embed into



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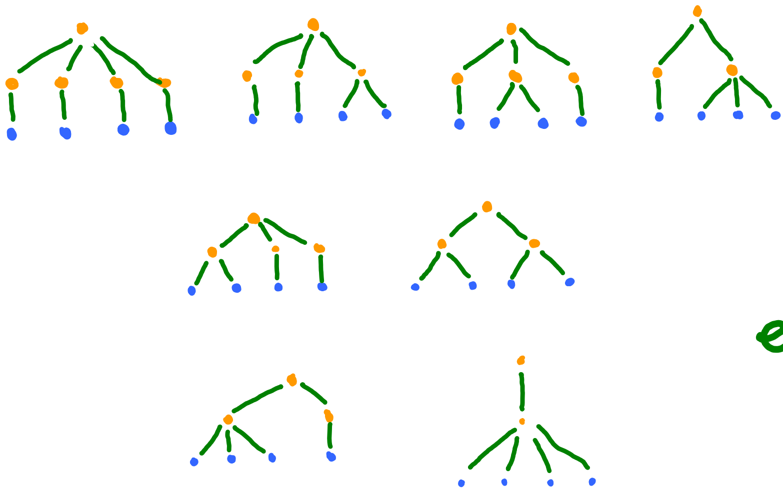
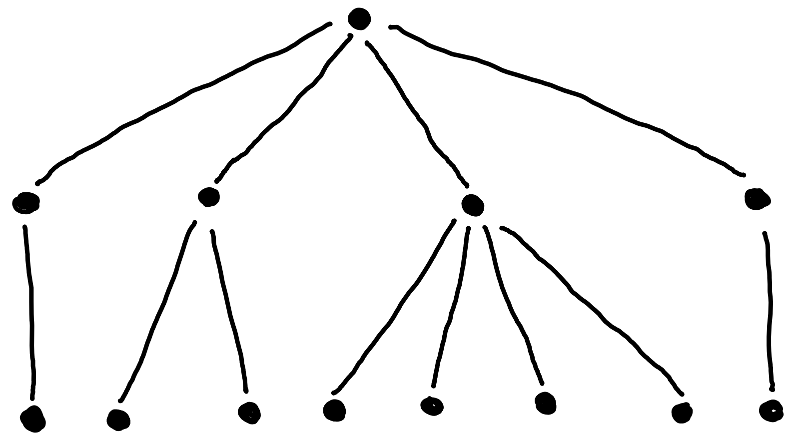
UNIVERSAL TREES



all embed into

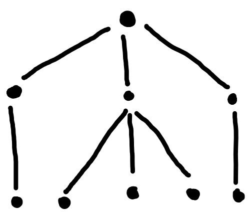


Leaves: 4
height: 2

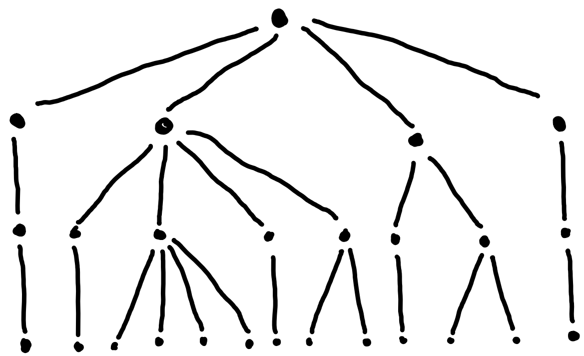
(4, 2)-universal tree

Universal trees

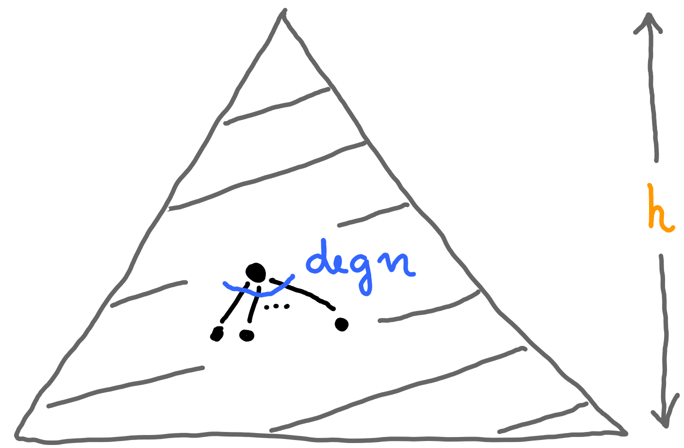
Definition T is an (n, h)-universal tree if every tree with n leaves and of height h embeds into T



$(3, 2)$-universal



$(4, 3)$-universal



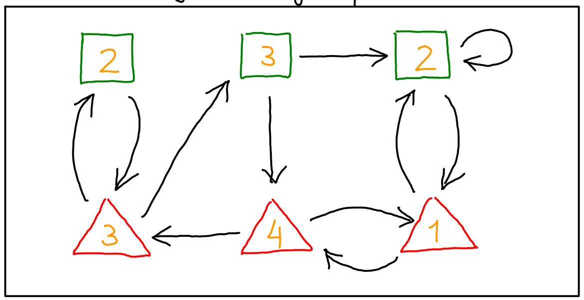
(n, h)-universal

$\Theta(n^h)$ size

PARITY GAMES

$n = |V|$

A game graph



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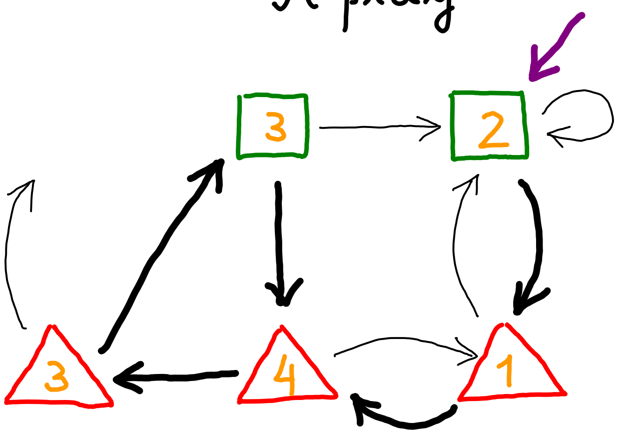


Even wins $\langle p_1, p_2, p_3, \dots \rangle$

iff

$\left[\limsup_{i \to \infty} p_i\right]$ is even

iff

all (but finitely many) occurrences of an odd priority are followed by an occurrence of a higher even priority

HIERARCHICAL MULTI-COUNTER $\mathcal{M}_{n,d}$

[Bernet, Janin, Walukiewicz 2003]

- **States**: multi-counters $(c_{d-1}, c_{d-3}, \dots, c_3, c_1)$ s.t. $0 \leq c_p \leq n$
- **Initial state**: $(0, \dots, 0)$
- **Transitions**:

$$(c_{d-1}, \dots, c_{p+2}, \mathbf{c_p}, c_{p-2}, \dots, c_1) \xrightarrow{p} \textbf{reject} \quad \text{if } c_p = n$$

$$(c_{d-1}, \dots, c_{p+2}, \mathbf{c_p}, c_{p-2}, \dots, c_1) \xrightarrow{p} (c_{d-1}, \dots, c_{p+2}, \mathbf{c_p}+1, 0, \dots, 0)$$

$$(c_{d-1}, \dots, c_{p+2}, \mathbf{c_p}, c_{p-2}, \dots, c_1) \xrightarrow{p-1} (c_{d-1}, \dots, c_{p+2}, \mathbf{c_p}, 0, \dots, 0)$$

FACT If $\mathcal{M}_{n,d}$ is "run" on an even graph with $\leq n$ vertices then no counter ever exceeds n.

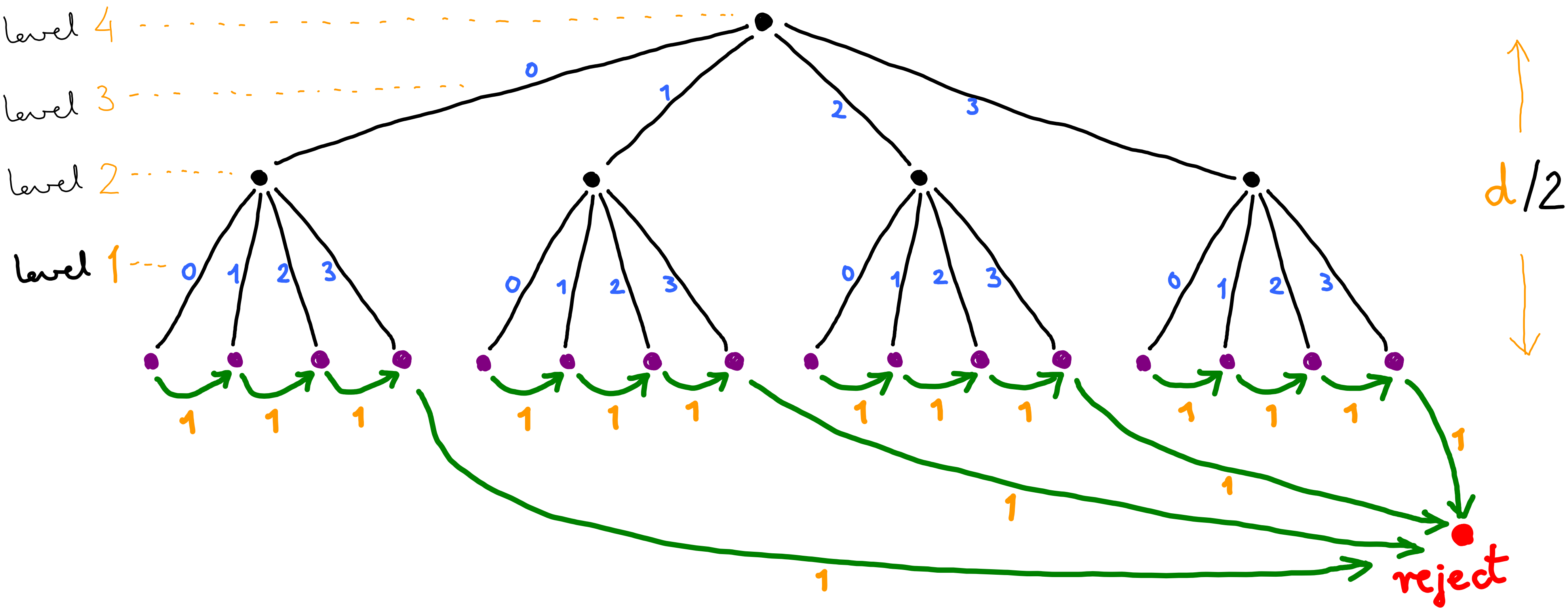
AN $n^{\frac{d}{2}+o(1)}$ ALGORITHM

$$\mathcal{M}_{n,d} = \{(c_{d-1}, c_{d-3}, \ldots, c_3, c_1) : 0 \leqslant c_p \leqslant n\}$$

<u>FACT</u> Parity game G
and safety game $G \rhd \mathcal{M}_{n,d}$
have the same winners

Multi-counters as an ordered tree

$\mathcal{M}_{3,4}$ — complete 4-ary tree of height 2



MULTI-COUNTERS AS AN ORDERED TREE

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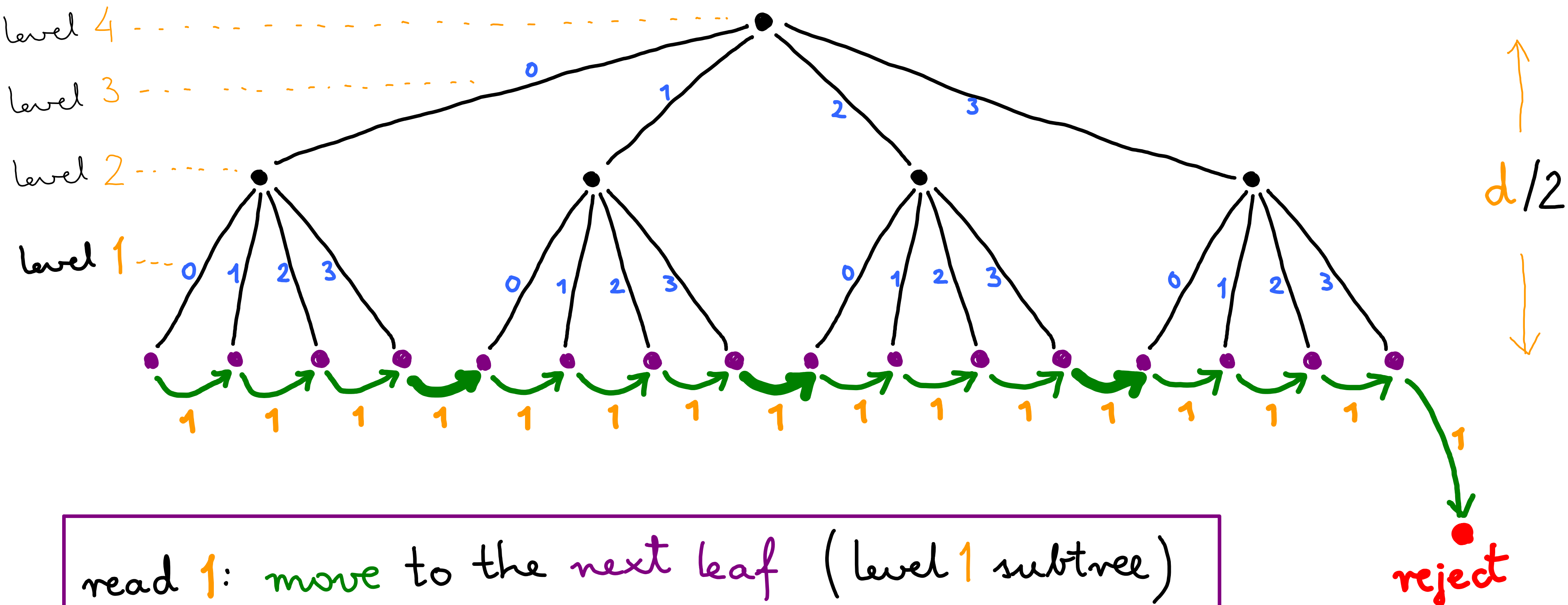

read 1: move to the next leaf (level 1 subtree)

MULTI-COUNTERS AS AN ORDERED TREE

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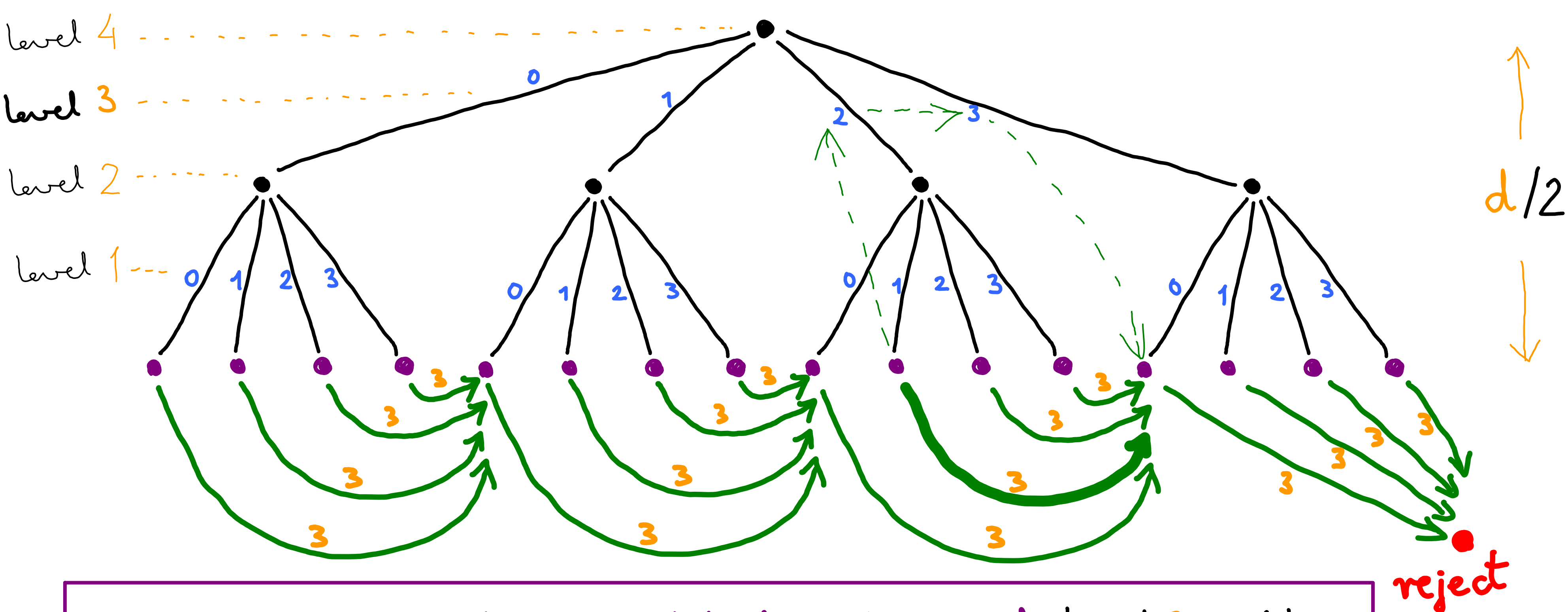
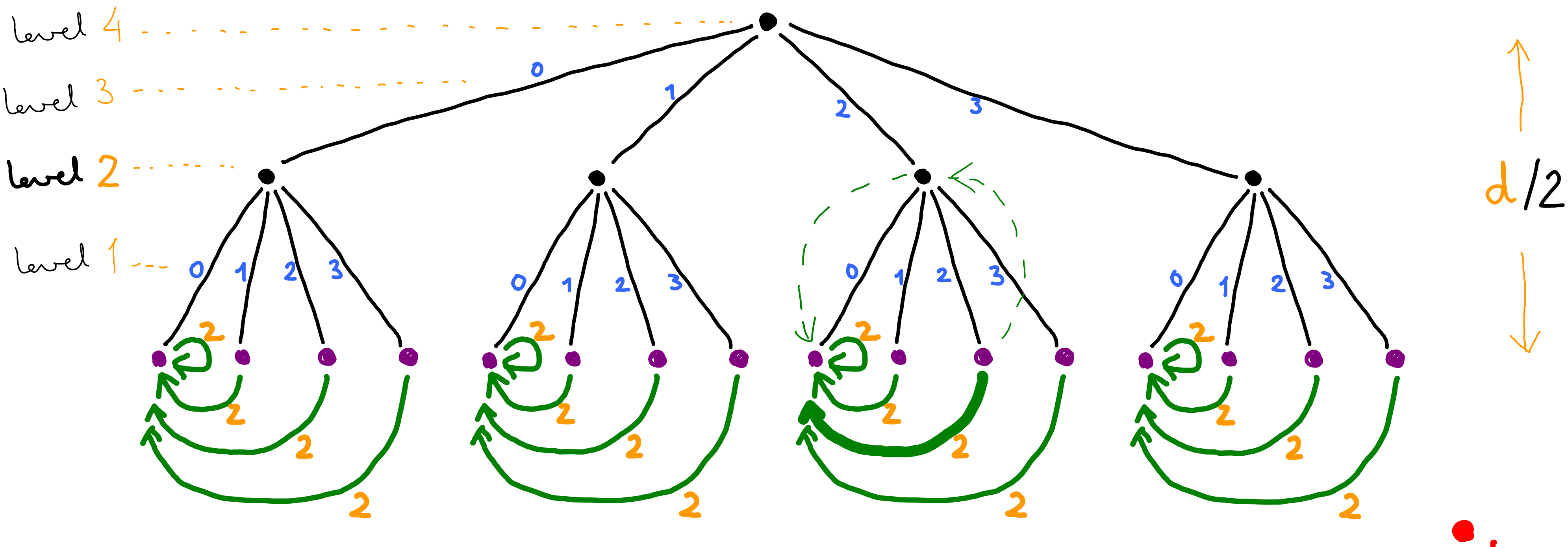


read 3: move to the smallest leaf in the next level 3 subtree

MULTI-COUNTERS AS AN ORDERED TREE

$\mathcal{M}_{3,4}$ — complete 4-ary tree of height 2



read 2: move to the smallest leaf in the same level 2 subtree

MULTI-COUNTERS AS AN ORDERED TREE

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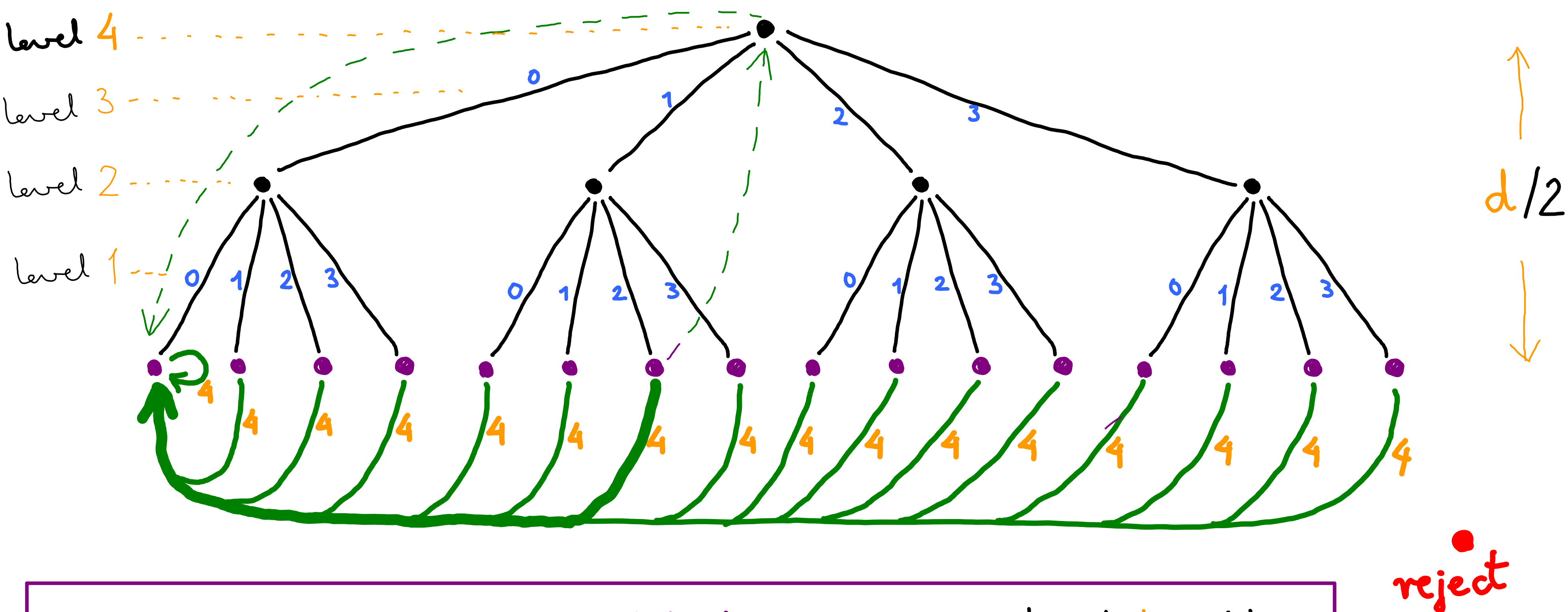

read 4: move to the smallest leaf in the same level 4 subtree

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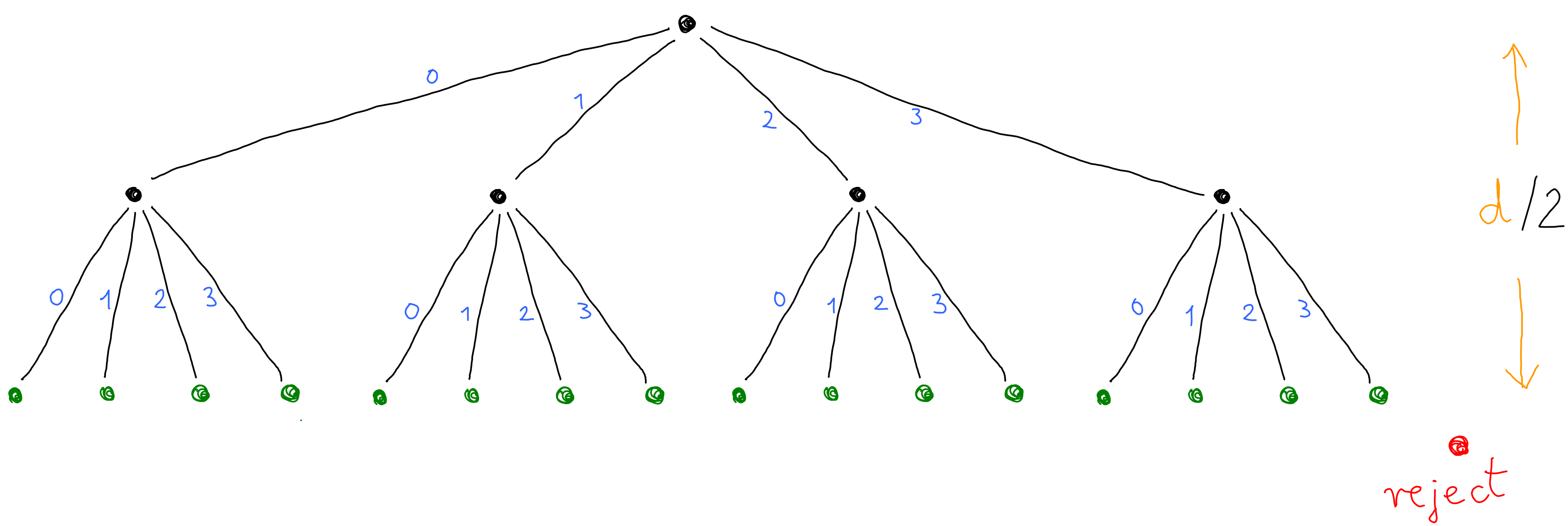
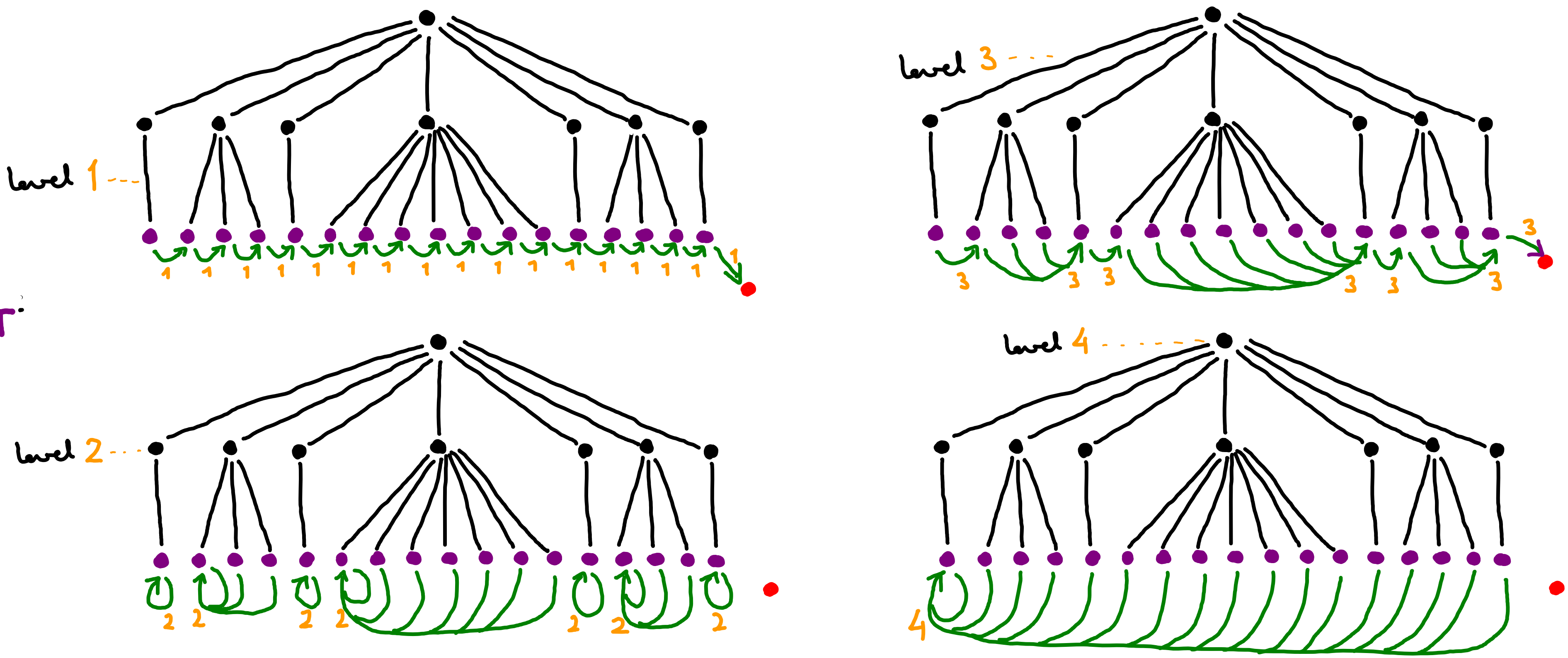


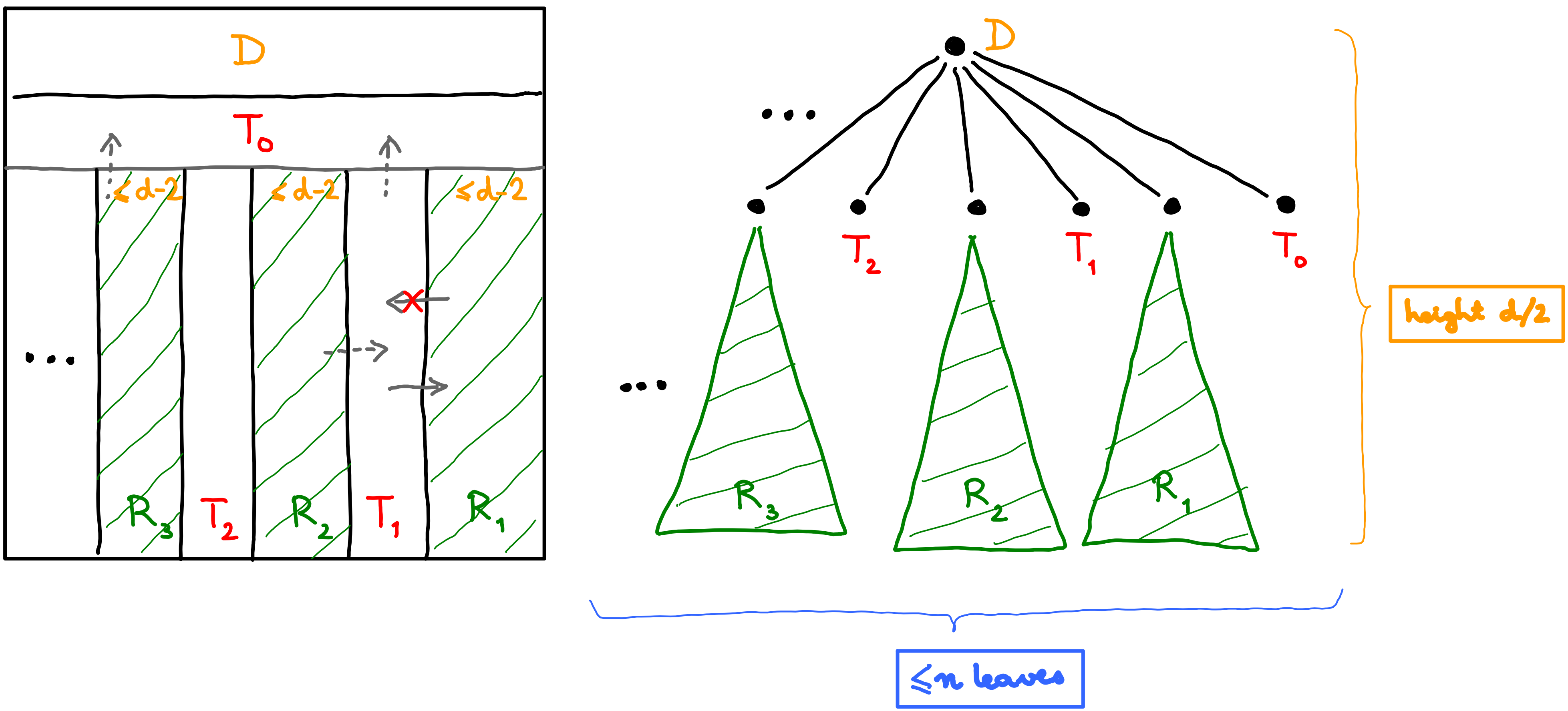
Question: Do we really need the complete $(n+1)$-ary tree of height $\frac{d}{2}$?

MULTI-COUNTERS FROM ANY ORDERED TREE

$\mathcal{M}_T$:



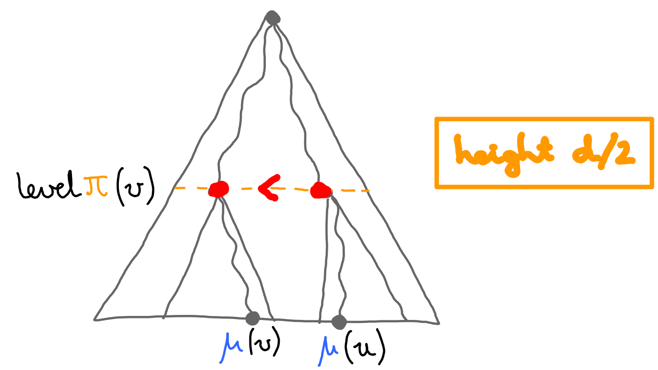
HIERARCHICAL DECOMPOSITION



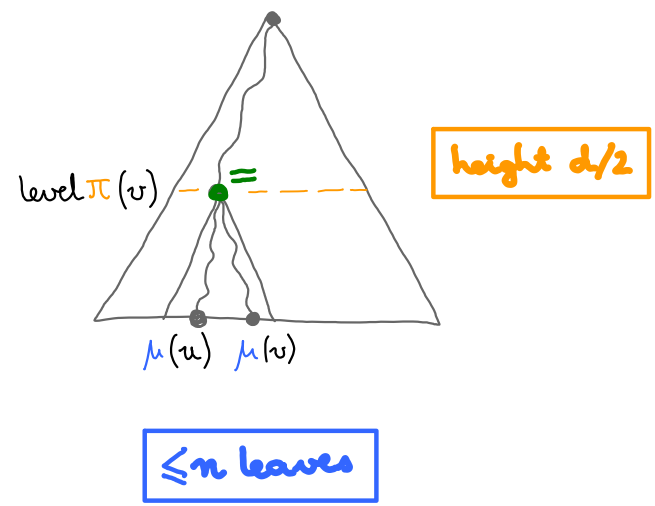
PROGRESS MEASURES

DEFINITION $\mu : V \to J$ is a progress measure for graph G if for every edge $(v,u) \in E$,

- $\boxed{\mu(v)|_{\pi(v)} < \mu(u)|_{\pi(v)}}$ if $\pi(v)$ is odd



- $\boxed{\mu(v)|_{\pi(v)} \leq \mu(u)|_{\pi(v)}}$ if $\pi(v)$ is even



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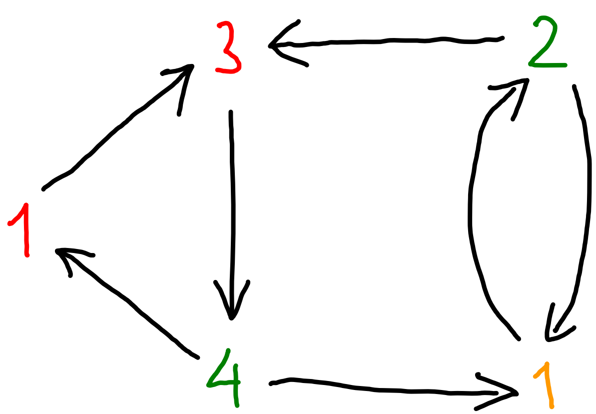

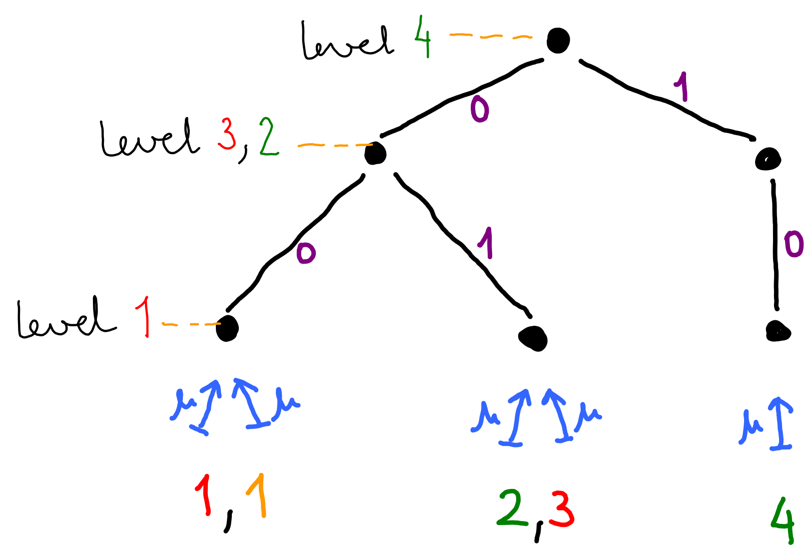

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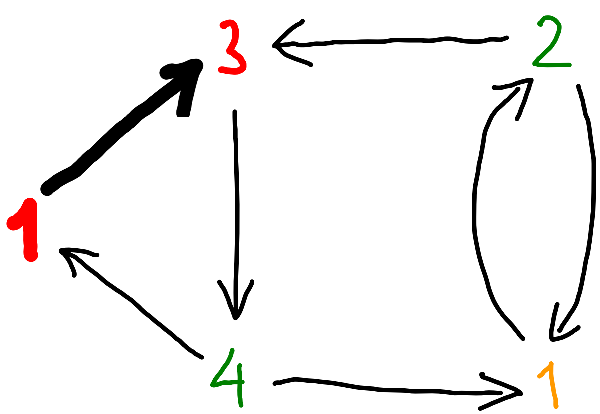

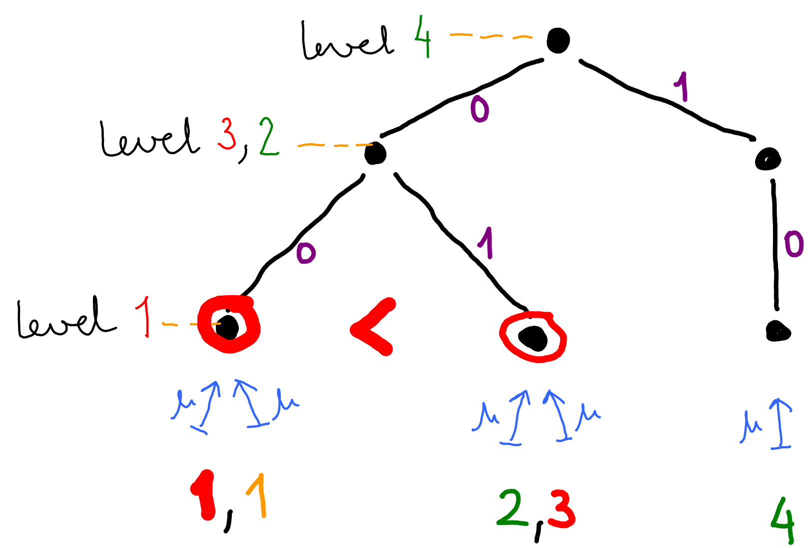

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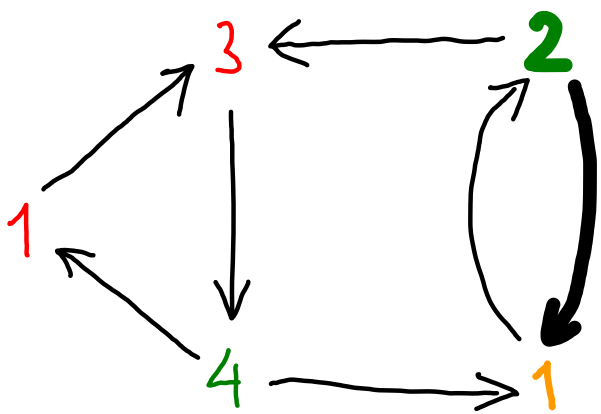

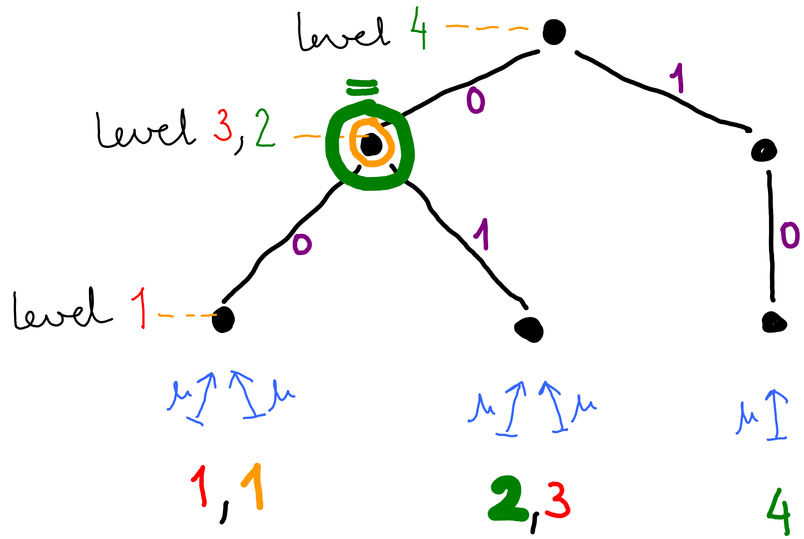

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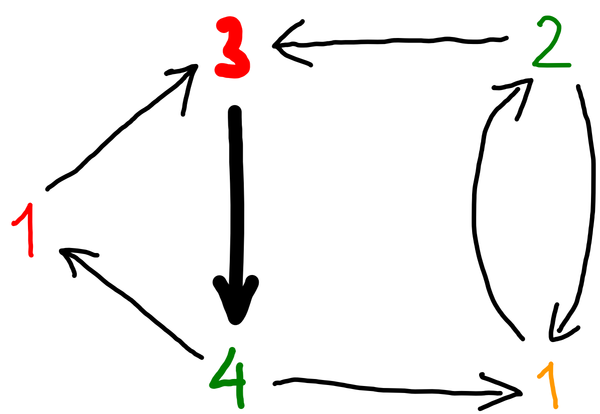
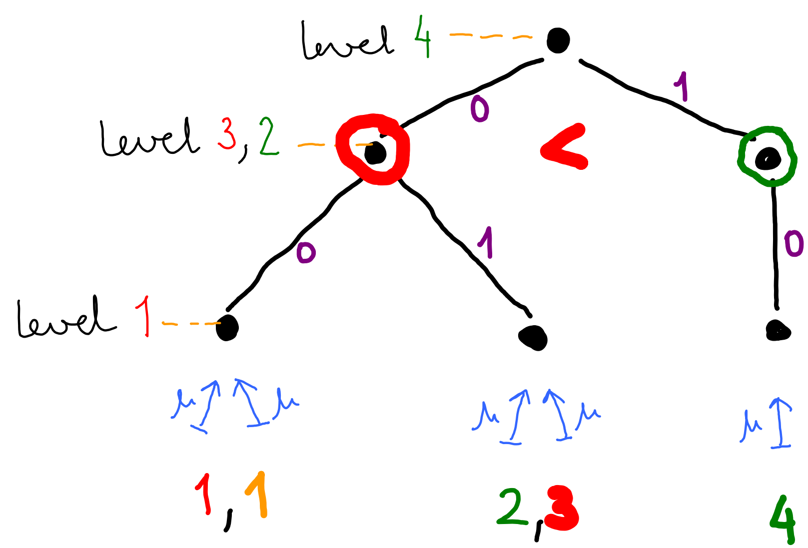




A "FUNDAMENTAL THEOREM" OF PARITY GAMES

THEOREM [...; Emerson, Jutla 1991; ...]
G has a progress measure iff every cycle in G is even

height d/2

≤n leaves

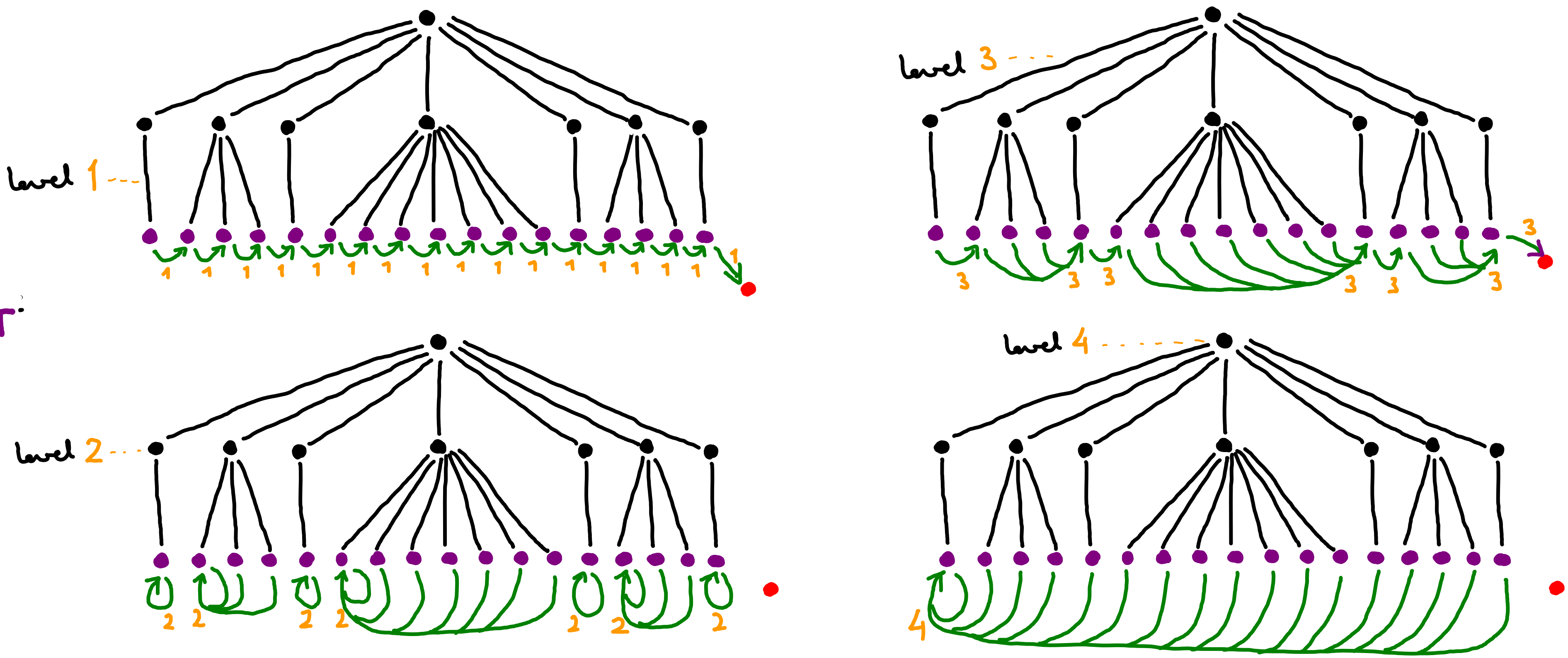
THE MULTI-COUNTER METHOD

$$\mathcal{M}_{n,d} = \{ (c_{d-1}, c_{d-3}, \dots, c_3, c_1) : \; 0 \leqslant c_p \leqslant n \}$$

FACT Parity game G
and safety game $G \triangleright \mathcal{M}_{n,d}$
have the same winners

MULTI-COUNTERS FROM ANY ORDERED TREE

$\mathcal{M}_T$:



THE MULTI-COUNTER METHOD

THM If $\mathcal{S} = \mathcal{M}_T$ where T is

- the complete n-ary tree of height $\frac{d}{2}$

then Parity game G

and **safety** game $G \triangleright \mathcal{S}$

have the same winners

THE MULTI-COUNTER METHOD

THM If $\mathcal{S} = \mathcal{M}_T$ where T is

- the complete n-ary tree of height $\frac{d}{2}$
- the tree of a progress measure $\boxed{\text{height } d/2}$ $\boxed{\leq n \text{ leaves}}$

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THE MULTI-COUNTER METHOD

THM If $\mathcal{S} = \mathcal{M}_T$ where T is

- the complete n-ary tree of height $\frac{d}{2}$
- the tree of a progress measure $\boxed{\text{height } d/2}$ $\boxed{\leq n \text{ leaves}}$
- an $(n, \frac{d}{2})$-universal tree

then Parity game G
and safety game $G \triangleright \mathcal{S}$
have the same winners

UNIVERSAL TREES: A QUASI-POLYNOMIAL UPPER BOUND

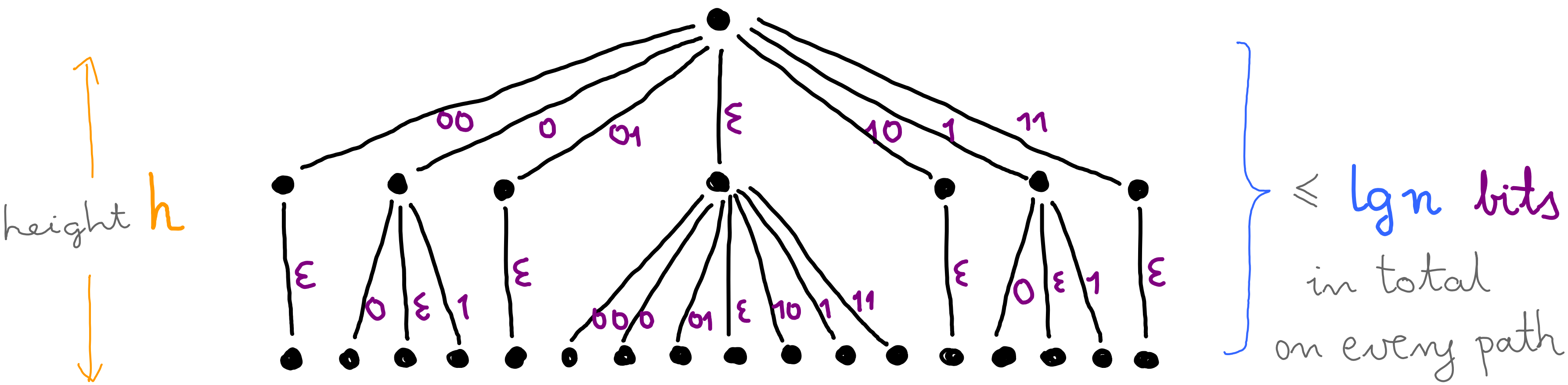
THEOREM [J., Lazić 2017]

There is an (n, h)-universal tree of size $n \cdot \binom{\lg n + h}{\lg n}$

- $\mathrm{poly}(n)$ if $h = O(\log n)$
- $n^{\lg\left(\frac{d}{\lg n}\right) + O(1)}$ if $h = \omega(\log n)$

UNIVERSAL TREES: A QUASI-POLYNOMIAL UPPER BOUND

(4,2)-universal ordered tree



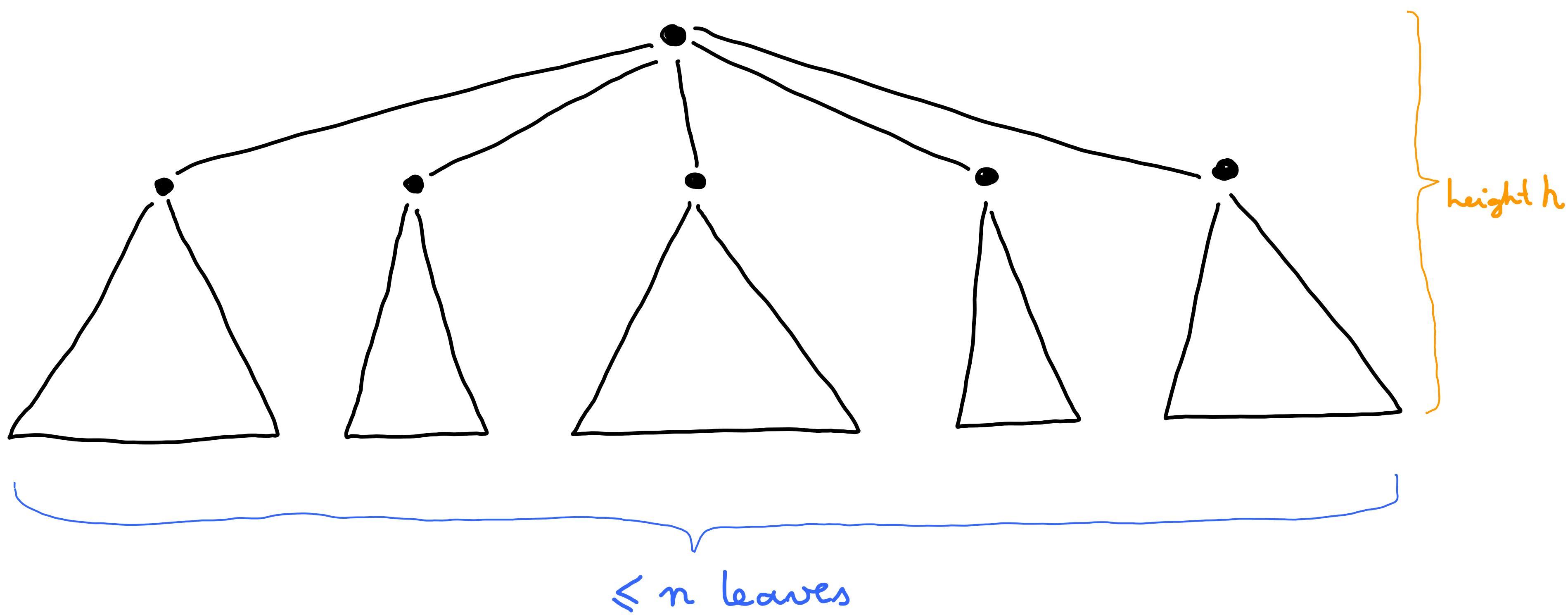
$$\boxed{00 \prec 0 \prec 01 \prec \varepsilon \prec 10 \prec 1 \prec 11}$$

the linear order on **bit strings** induced by $0 \prec \varepsilon \prec 1$

Tree Coding Lemma

Lemma

If an ordered tree T has height h and $\leq n$ leaves then there is an order-preserving labelling of T by $(\{0,1\}^*, <)$ s.t. on every path (from root to leaf) $\leq \lg n$ bits are used



TREE CODING LEMMA

LEMMA

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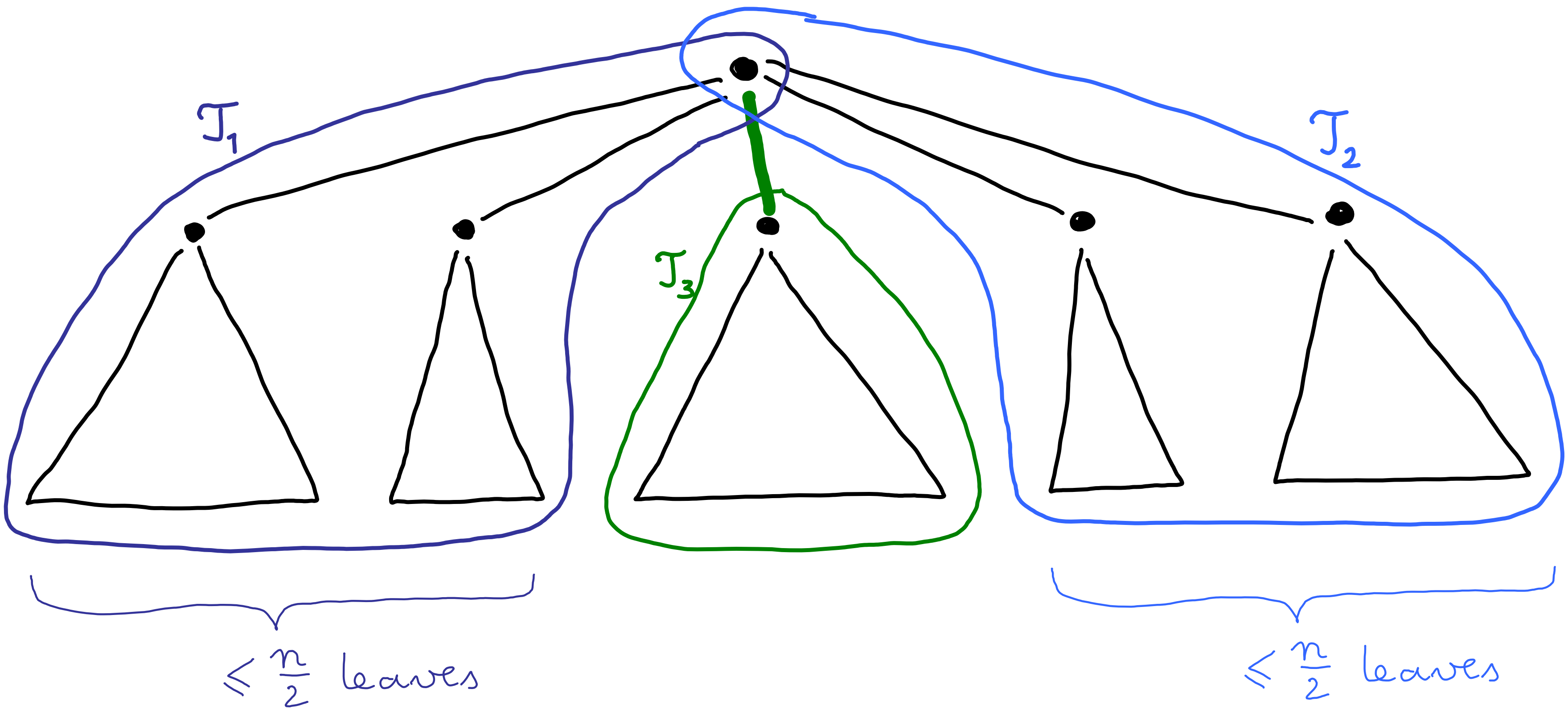

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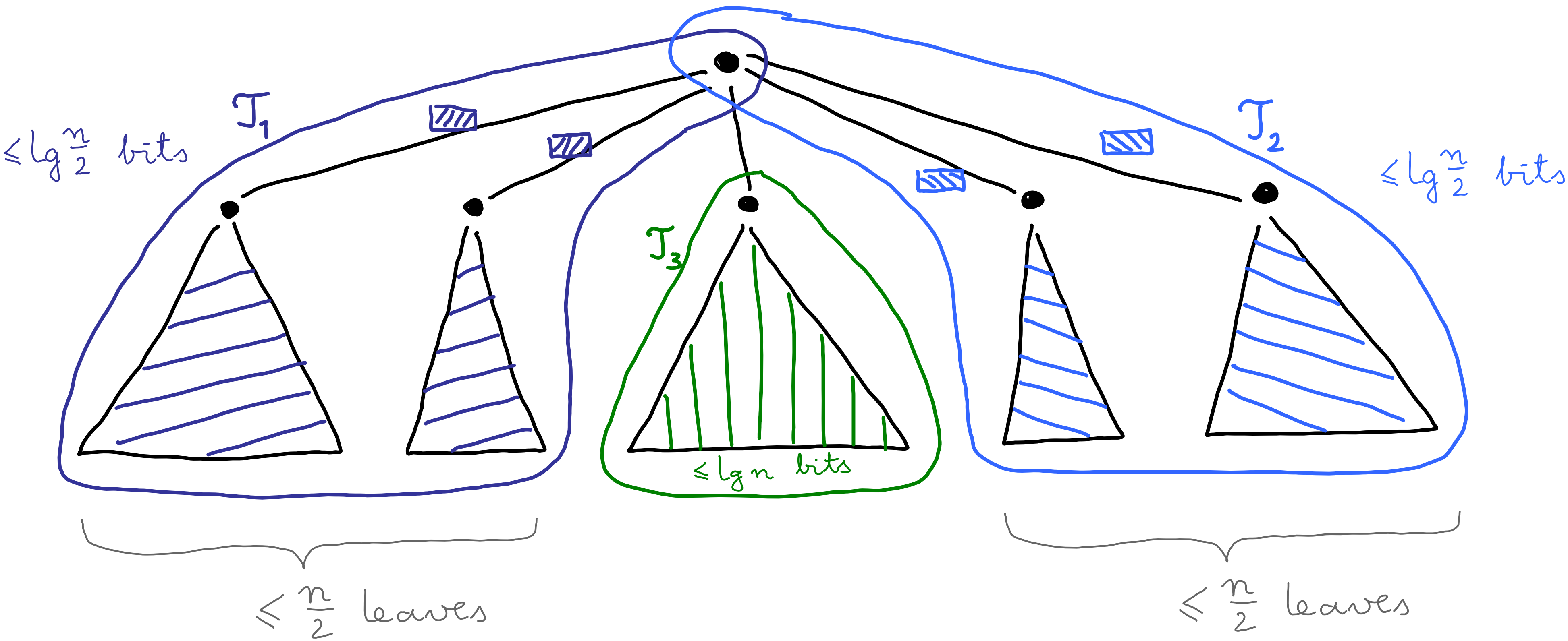
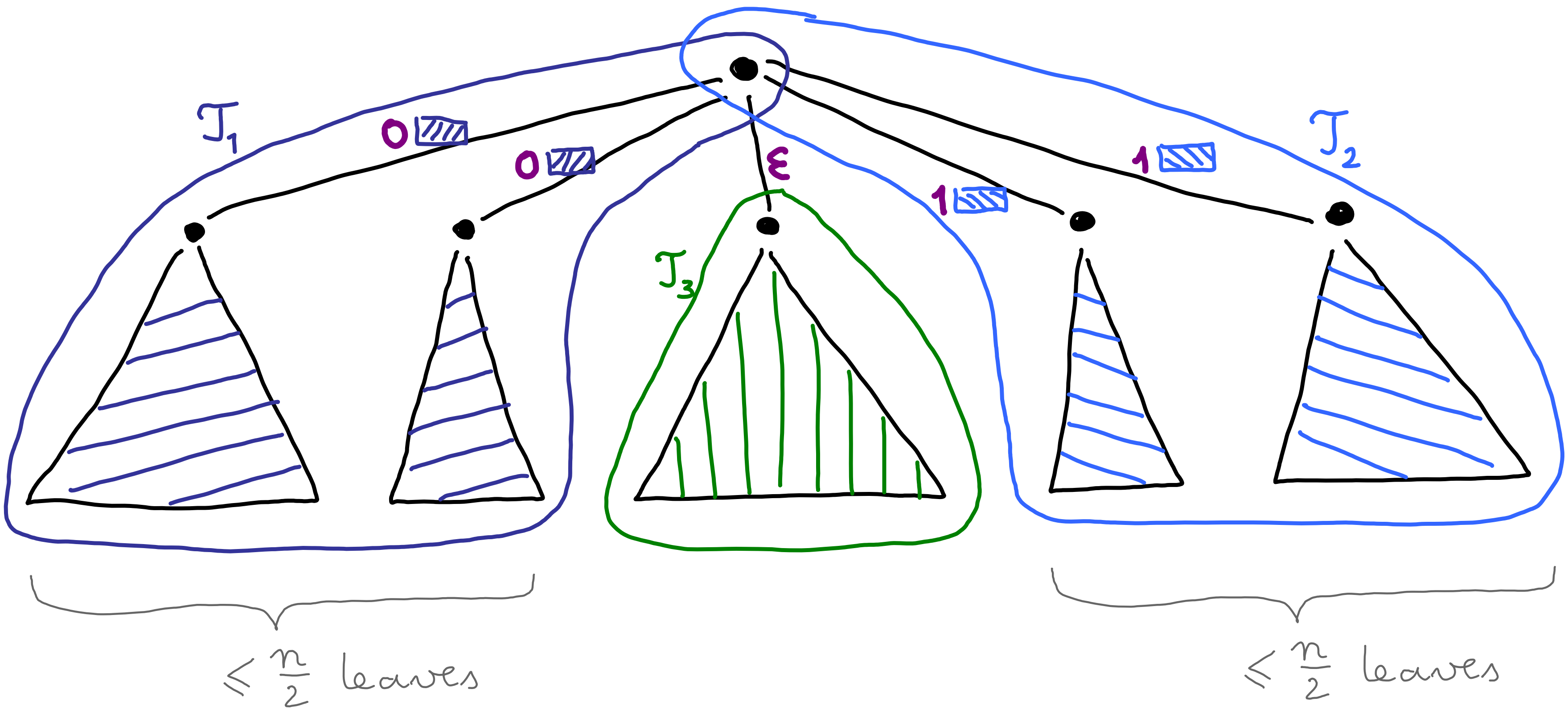


TREE CODING LEMMA

LEMMA

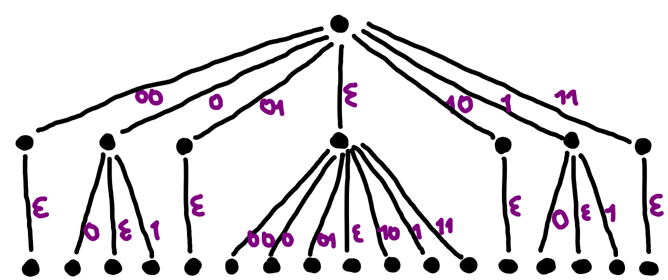
If an ordered tree $\mathcal{T}$ has height h and $\leqslant n$ leaves then there is an order-preserving labelling of $\mathcal{T}$ by $(\{0,1\}^*, <)$ s.t. on every path (from root to leaf) $\leqslant \lg n$ bits are used



SUCCINCT MULTI-COUNTERS

$$S_{n,h} \stackrel{\text{df}}{=} \left\{ \langle s_h, s_{h-1}, \dots, s_1 \rangle : s_i \in \{0,1\}^* \text{ and } \sum_{i=1}^{h} |s_i| \leqslant \lceil \lg n \rceil \right\}$$

$S_{4,2}$



$$\boxed{00 \prec 0 \prec 01 \prec \varepsilon \prec 10 \prec 1 \prec 11}$$

bits sufficient to represent a succinct multi-counter

Fact $$|S_{n,h}| \leqslant 2^{\overbrace{\lceil \lg n \rceil \cdot (1 + \lceil \lg h \rceil)}} = \boxed{n^{\lg h + O(1)}}$$

For each bit,

the bit

the co-ordinate the bit belongs to

THE SIZE OF $S_{n,h}$

- $|S_{n,h}| \leq 2^{\lceil \lg n \rceil} \cdot \binom{\lceil \lg n \rceil + h}{h}$

- $$|S_{n,h}| = \begin{cases} O(n \cdot \lg^h n) & \text{if } h = O(1) \\ O\left(n^{1+o(1)}\right) & \text{if } h = o(\log n) \\ \tilde{\Theta}\left(n^{\lg(\delta+1) + \underbrace{\lg(e_\delta)+1}_{\leq 3.72}}\right) & \text{if } h = \lceil \delta \cdot \lg n \rceil \\ O\left(h \cdot n^{\lg\left(\frac{h}{\lg n}\right) + 0.45}\right) & \text{if } h = \omega(\log n) \end{cases}$$

where $e_\delta = \left(1 + \frac{1}{\delta}\right)^\delta$

SMALLEST UNIVERSAL TREES ARE QUASI-POLYNOMIAL

THEOREM [J., Lazić 2017]

There is an (n, h)-universal tree of size $n \cdot \binom{\lg n + h}{h} = n^{\lg\left(\frac{h}{\lg n}\right) + O(1)}$

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2019]

Every (n, h)-universal tree is of size at least $\binom{\lg n + h - 1}{h - 1} \geq n^{\lg\left(\frac{h}{\lg n}\right) - 1}$

SMALLEST UNIVERSAL TREES ARE QUASI-POLYNOMIAL

$L(n,h) \overset{df}{=}$ smallest # leaves in an (n,h)-universal tree

$$\boxed{\begin{array}{l} L(n,1) = n \\ L(1,h) = 1 \end{array}}$$

Upper bound recurrence $\boxed{L(n,h) \leq L(n,h-1) + 2 \cdot L\left(\frac{n}{2}, h\right)}$

Lower bound recurrence $\boxed{L(n,h) \geq L(n,h-1) + L\left(\frac{n}{2}, h\right)}$

<u>COROLLARY</u> $\boxed{L(n,h) \geq \binom{\lg n + h - 1}{h-1}}$

SOLVING PARITY GAMES USING SAFETY AUTOMATA

[Bernet, Janin, Walukiewicz 2003]

parity game

deterministic **safety** automaton

the **"chained product"** game is played on G

$$\underbrace{G \rhd A}_{\text{safety game}}$$

- A reads **priority sequences**
- A declares Odd the winner if **REJECT** state is reached

SOLVING PARITY GAMES USING SAFETY AUTOMATA

[Bernet, Janin, Walukiewicz 2003]

parity game

deterministic **safety** automaton

$$G \triangleright A$$

[Calude, Jain, Khoussainov, Li, Stephan 2017]

QUESTION What properties of A make G and $G \triangleright A$ have the same winners?

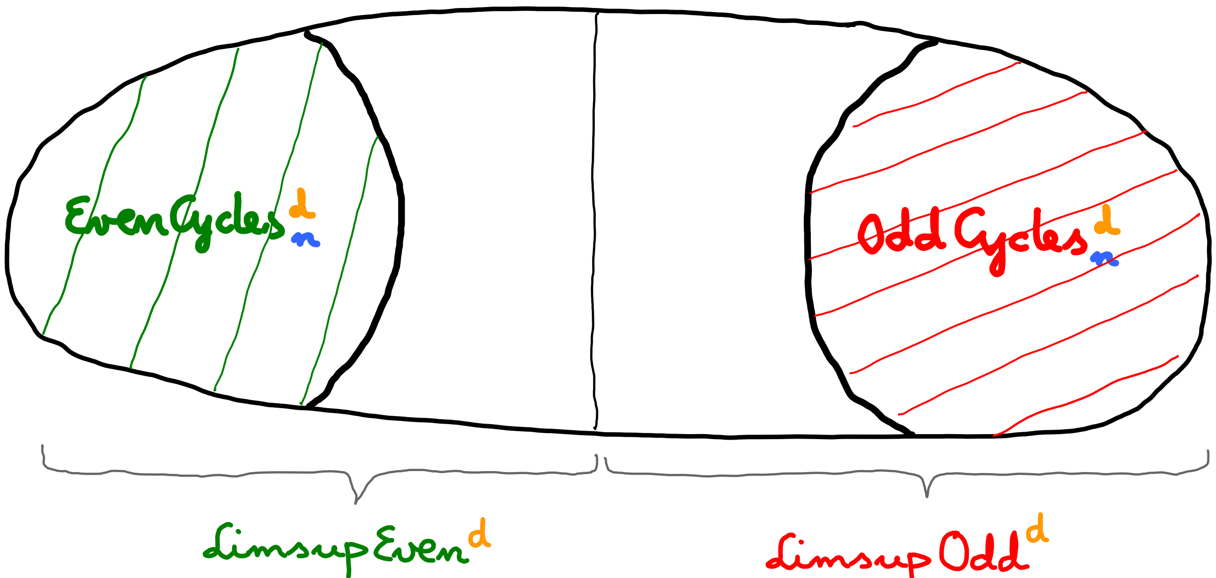
- If Even plays a positional winning strategy in G then A **accepts**
- If Odd plays a positional winning strategy in G then A **rejects**

LANGUAGES OF PRIORITY SEQUENCES

- $\mathrm{limsupEven}^d \subseteq \{1, 2, \ldots, d\}^\omega$: won by Even

LANGUAGES OF PRIORITY SEQUENCES

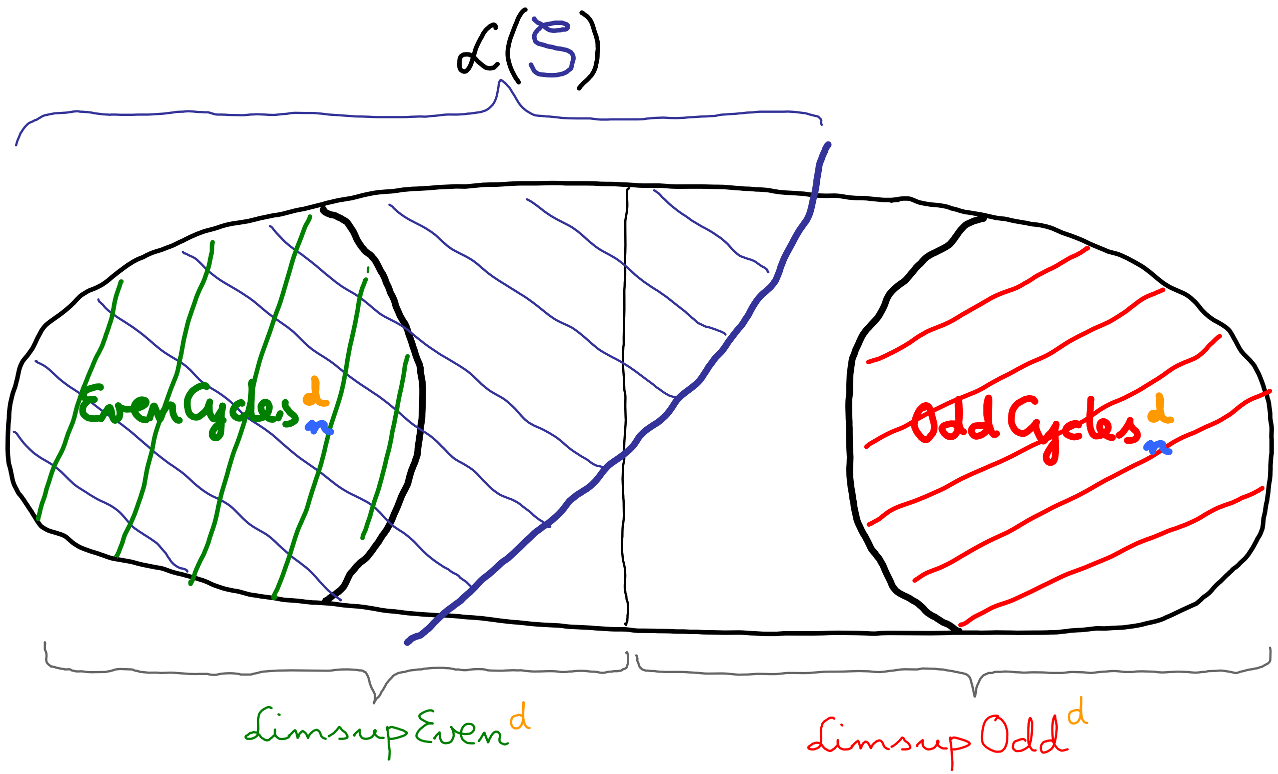
- $\text{LimsupEven}^d \subseteq \{1,2,\ldots,d\}^\omega$: won by Even
- $\text{EvenCycles}^d_n \subseteq \{1,2,\ldots,d\}^\omega$: arising from a game graph (with $\leq n$ vertices and $\leq d$ priorities) in which all cycles are even



SEPARATING (SAFETY) AUTOMATA

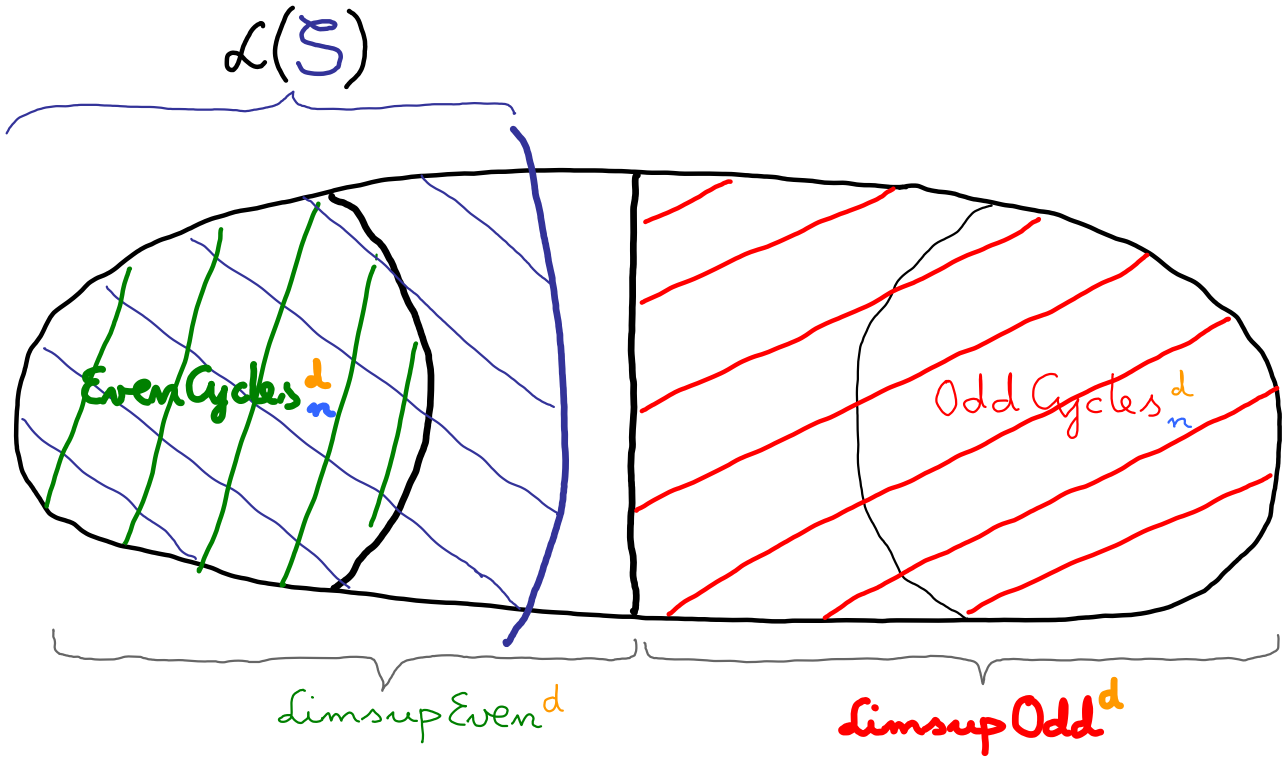
[Bojańczyk, Czerwiński 2018]

DEFINITION A finite (safety) automaton $\mathcal{S}$ is an (n,d)-separator if $\mathcal{L}(\mathcal{S})$ separates EvenCycles^d_n from OddCycles^d_n



SEPARATING (SAFETY) AUTOMATA

DEFINITION A finite (safety) automaton $\mathcal{S}$ is a strong (n, d)-separator if $\mathcal{L}(\mathcal{S})$ separates $\mathrm{EvenCycles}_n^d$ from $\mathrm{LimsupOdd}^d$



THE SEPARATION METHOD

FACT If S is an (n,d)-separator and the parity game G has $\leqslant n$ vertices and $\leqslant d$ priorities then games G and $G \triangleright S$ have the same winners

parity game

safety game

A QUASI-POLYNOMIAL SEPARATOR

THEOREM [Calude et al. 2017; Bojańczyk, Czerwiński 2019]

There is a strong (n, d)-separator of size $n^{\lg d + O(1)}$

- **States**: "play statistics" $(p_{\lg n}, p_{\lg n - 1}, \dots, p_1, p_0) \in \{0, 1, 2, \dots, d\}^{\lg n + 1}$

- **Transitions**: see [Calude et al. 2017] or [Bojańczyk, Czerwiński 2018]

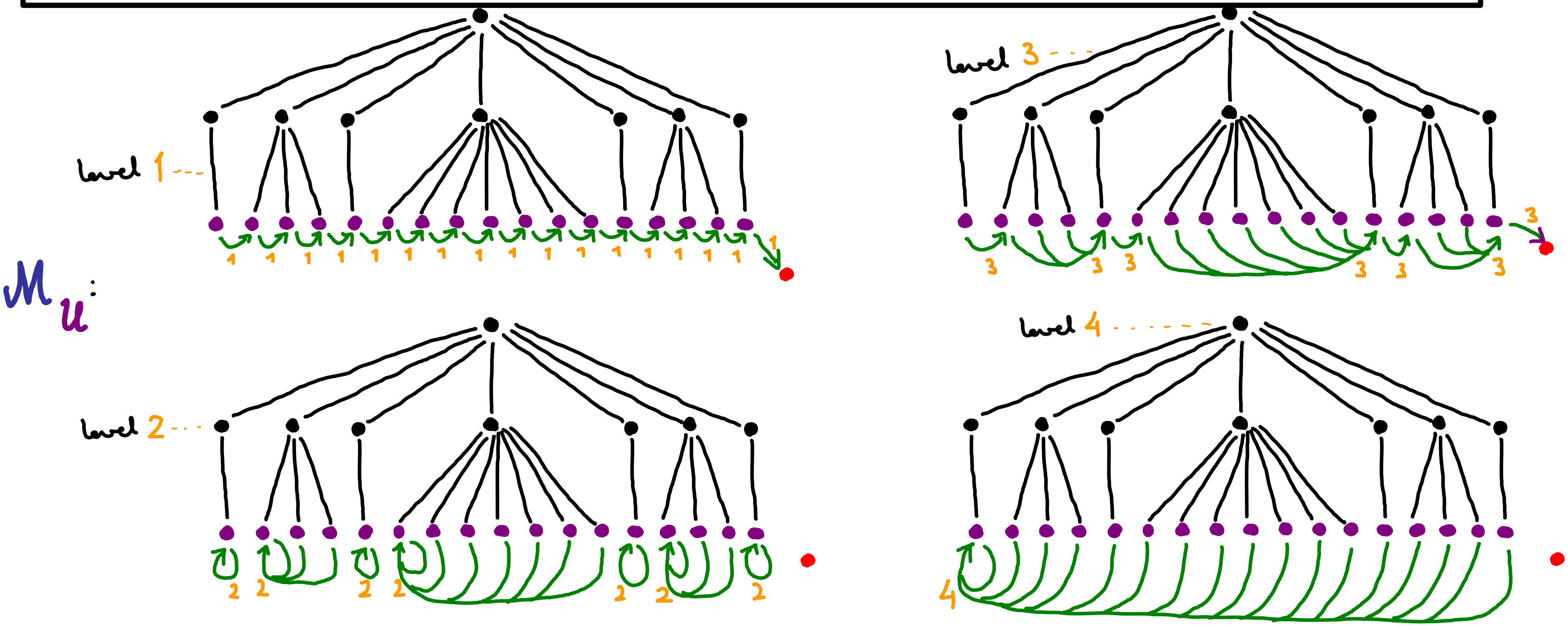
$\lg d \cdot \lg n$-space **TM**

automaton of size $n^{\lg d + O(1)}$

SEPARATING AUTOMATA FROM UNIVERSAL TREES

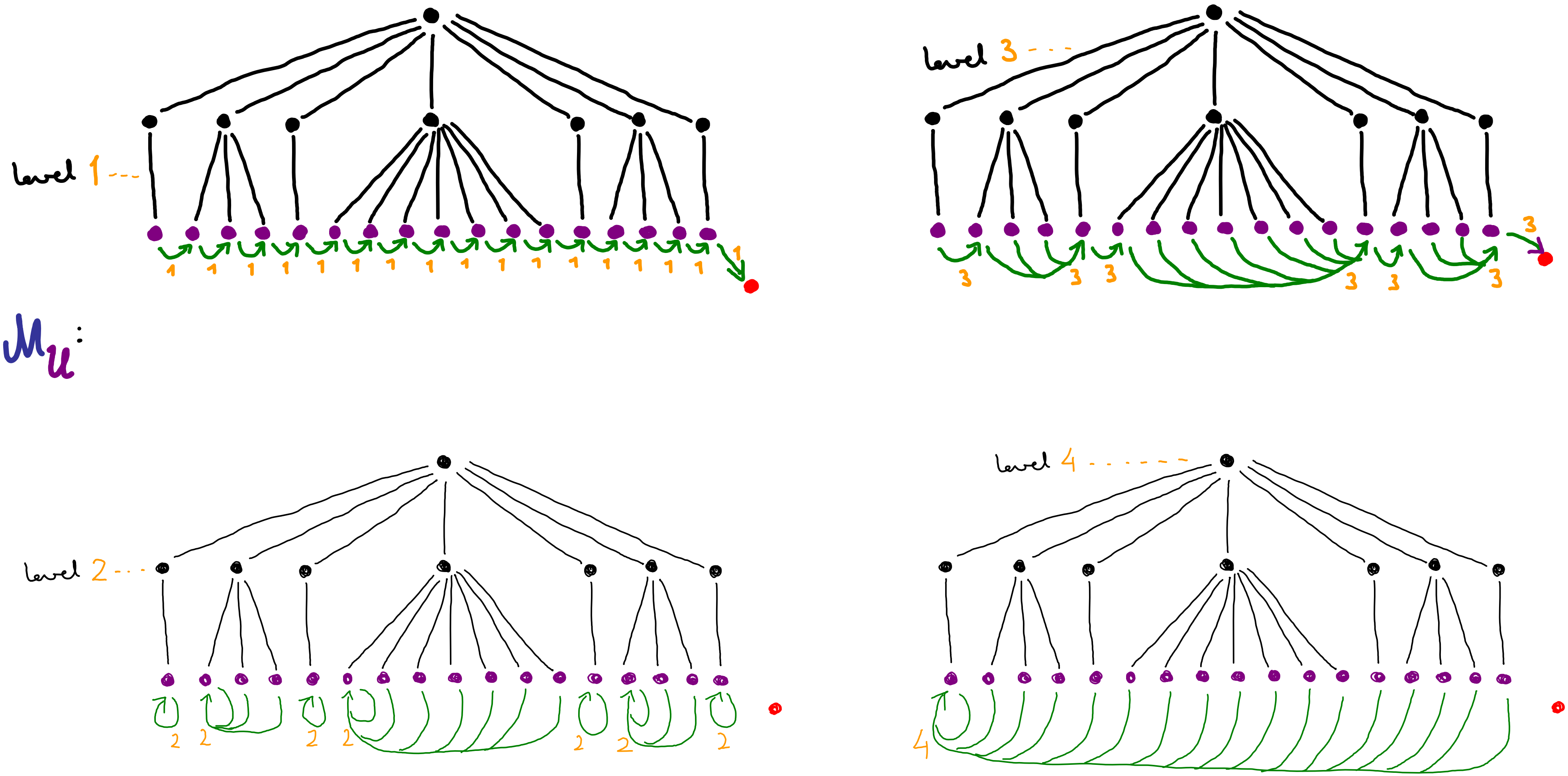
THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2019]

Leaves of every $(n, \frac{d}{2})$-universal tree are the states of a strong (n, d)-separator



$\mathcal{M}_u$ REJECTS ALL WORDS IN Odd_d

$\mathcal{M}_u$:



$\mathcal{M}_{\mathcal{U}}$ ACCEPTS ALL WORDS IN $\mathrm{Even}_{n,d}$

- If all cycles in G are even then there is a progress measure

$$\mu : V \longrightarrow \mathcal{T} \hookrightarrow \mathcal{U}$$

- Let $\mathcal{M}_{\mathcal{U}}$ run on G:

Invariant: $\boxed{\text{when } \mathcal{M}_{\mathcal{U}} \text{ visits vertex } v, \text{ its state is } \leqslant \mu(v)}$

SEPARATING AUTOMATA FROM REGISTER GAMES

THEOREM [Lehtinen 2018]

There is parity strong (n,d)-separator with $\lg n$ priorities and of size $n \cdot d^{\lg n} = n^{\lg d + O(1)}$

COROLLARY

Parity games can be solved in time

$$\left(n^{\lg d}\right)^{\lg n}$$

UNIVERSAL TREES ≡ SEPARATING AUTOMATA

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2019]

Leaves of every $(n, \frac{d}{2})$-universal tree
are the states of a strong (n, d)-separator

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2019]

States in every strong (n, d)-separator
include all the leaves in an $(n, \frac{d}{2})$-universal tree

UNIVERSAL TREES AND SEPARATING AUTOMATA ARE QUASI-POLYNOMIAL

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

The sizes of smallest universal trees
and of smallest separating automata
are quasi-polynomial

UNIVERSAL TREES ≡ SEPARATING AUTOMATA

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

Leaves of every $(n, \frac{d}{2})$-universal tree are the states of a strong (n, d)-separator

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

States in every strong (n, d)-separator include all the leaves in an $(n, \frac{d}{2})$-universal tree

SMALLEST UNIVERSAL TREES ARE QUASI-POLYNOMIAL

THEOREM [J., Lazić 2017]

There is an $(n, \frac{d}{2})$-universal tree of size $n \cdot \binom{\lg n + \frac{d}{2}}{\lg n} = n^{\lg\left(\frac{d}{\lg n}\right) + O(1)}$

THEOREM [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

Every $(n, \frac{d}{2})$-universal tree is of size at least $\binom{\lg n + \frac{d}{2} - 2}{\lg n - 1} \geq n^{\lg\left(\frac{d}{\lg n}\right) - 2}$

HOT OFF/ON THE PRESS

- [Daviaud, J., Lehtinen 2019]
  "Alternating weak automata from universal trees"

- [Daviaud, J., Thejaswini 20??]
  "The Strahler number of a parity game"

- [Parys 2019]
  "Zielonka's algorithm in quasi-polynomial time"

  [Lehtinen, Schewe, Wojtczak 2019]
  "Improving the complexity of Parys' recursive algorithm"

  [J., Morvan 20??]
  "A universal algorithm for parity games and model checking"

THE STRAHLER NUMBER

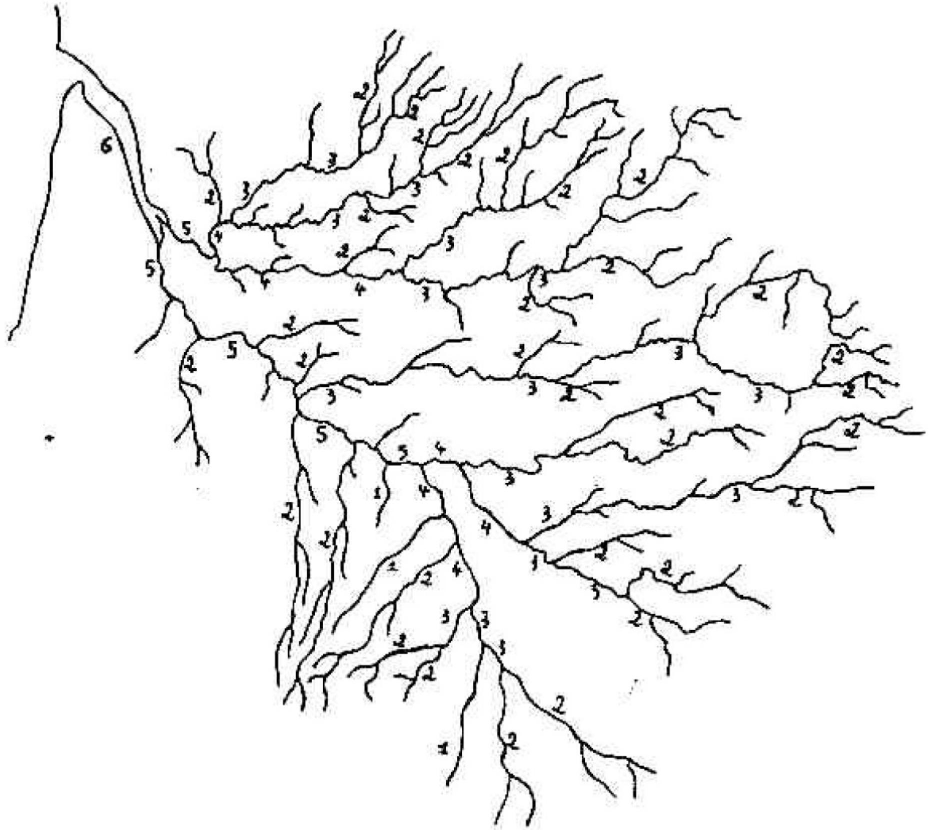


Fig. 2 : The Horton–Strahler rule for the Garonne river network

[Viennot 1990]

OPEN PROBLEMS

- Is there a (randomized) poly-time algorithm?

- Design separators that are smaller than strong separators

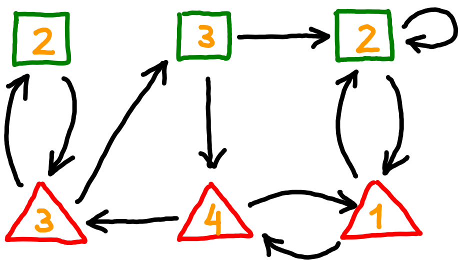
- Design parity separators that are smaller than safety separators

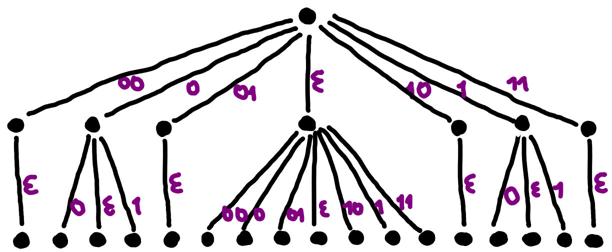
- Close the $\Omega(n \log n)$ and $n^{O(\log d)}$ gap for the blow-up in alternating parity to weak automata translation

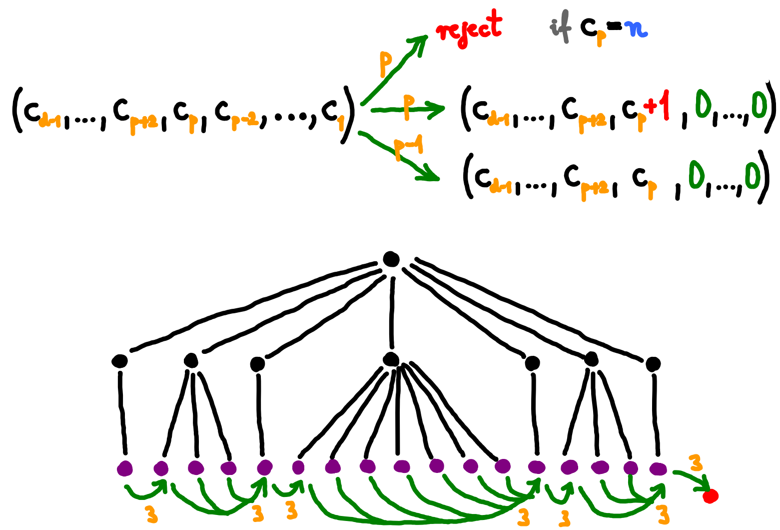
UNIVERSAL TREES

QUASI-POLYNOMIAL

PARITY GAMES SEPARATING AUTOMATA







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