UNIVERSAL TREES
QUASI-POLYNOMIAL
PARITY GAMES
SEPARATING AUTOMATA

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A game graph

\[
G = (V = V_{\text{even}} \cup V_{\text{odd}}, E, \pi)
\]

\[
\pi : V \rightarrow \{1, 2, 3, 4, 5, \ldots, d\}
\]
A game graph

\[ G = (V = V_{\text{even}} \cup V_{\text{odd}}, E, \pi) \]

\[ \pi : V \rightarrow \{1, 2, 3, 4, 5, \ldots, d\} \]

A play

Even wins \( \langle p_1, p_2, p_3, \ldots \rangle \) iff \[ \limsup_{i \rightarrow \infty} p_i \] is even
THEOREM [Emerson, Jutla 1991; Mostowski 1991; re-proved since 1960’s]

Parity games are positionally determined

COROLLARY [Emerson, Jutla, Sistla 1993]

Deciding the winner in parity games is in $\text{NP} \cap \text{co-NP}$
# Algorithms for Solving Parity Games

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Algorithms for solving parity games

- $d + O(1) \eta$
  [McNaughton 1993; Zielonka 1998]

- $d^2 + O(1) \eta$
  [Browne, Clarke, Jha, Long, Manreza 1994; Seidl 1996; J. 2000]

- $d + O(1) \eta$
  strategy iteration
  [Vöge, J. 2000]

- $\Omega(n)$
  [Friedmann 2009]

- $O(\sqrt{n}) \eta$

- $d^3 + O(1) \eta$
  [Scheur 2007]

- $\log d + O(1) \eta$
  [Calude, Jain, Khovsainov, Li, Stephan 2017; J., Lazar 2017; Lehtinen 2018]

- policy iteration for MDPs
  [Fearnley 2010]

- randomized simplex
  [Hansen, Friedmann, Zwick 2011]
Ordered trees

$\langle \mathbb{N}, < \rangle$

$\langle \mathbb{B}^*, <_{\text{lex}} \rangle$
Ordered trees

(\mathbb{N}, <)

\text{embeds into}

(\mathbb{B}^*, <_{\text{lex}})
Ordered trees

Does not embed into

\((\mathbb{N}, \leq)\)

\((\mathbb{B}^*, \leq_{lex})\)
Universal trees

Leaves: 4
Height: 2

(4, 2)-universal tree

All embed into
**Universal Trees**

**Definition**

$T$ is an $(n, h)$-universal tree if every tree with $n$ leaves and of height $h$ embeds into $T$.

$(3, 2)$-universal \hspace{1cm} $(4, 3)$-universal \hspace{1cm} $(n, h)$-universal

$\Theta(n^h)$ size
**Parity Games**

$$n = |V|$$

A game graph:

$$G = (V = V_{\text{even}} \cup V_{\text{odd}}, E, \pi)$$

$$\pi : V \rightarrow \{1, 2, 3, 4, 5, \ldots, d\}$$

A play:

**Even wins** \(\langle p_1, p_2, p_3, \ldots \rangle\)

iff

$$\left[ \limsup_{i \to \infty} p_i \right]$$

is even

iff

all (but finitely many) occurrences of an odd priority are followed by an occurrence of a higher even priority
Hierarchical multi-counter $M_{n,d}$

[Bennet, Jamum, Wahskeyence 2003]

- **States**: multi-counters $(C_{d-1}, C_{d-2}, \ldots, C_3, C_1)$ s.t. $0 \leq C_p \leq n$

- **Initial state**: $(0, \ldots, 0)$

- **Transitions**:
  
  $$(C_{d-1}, \ldots, C_{p+2}, C_p, C_{p-2}, \ldots, C_1)$$

  - $P$:
    
    $$(C_{d-1}, \ldots, C_{p+2}, C_p^+, C_{p-2}, \ldots, C_1)$$

  - $P-1$:
    
    $$(C_{d-1}, \ldots, C_{p+2}, C_p, C_{p-2}, \ldots, C_1)$$

  - **reject** if $C_p = n$

**FACT** If $M_{n,d}$ is "run" on an even graph with $\leq n$ vertices then no counter ever exceeds $n$. 
An $\frac{n}{2} + O(1)$ ALGORITHM

$$M_{n,d} = \{(c_{d-1}, c_{d-3}, \ldots, c_3, c_1) : 0 \leq c_p \leq n\}$$

**FACT** Parity game $G$
and safety game $G \triangleright M_{n,d}$
have the same winners
Multi-counters as an ordered tree

$M_{3,4}$ — complete 4-way tree of height 2
Multi-counters as an ordered tree

$M_{3,4}$ — complete 4-ary tree of height 2

Read 1: move to the next leaf (level 1 subtree)
Multi-counters as an ordered tree

\[ M_{3,4} \] — complete 4-ary tree of height 2

level 4
level 3
level 2
level 1

read 3: move to the smallest leaf in the next level 3 subtree

\[ \text{reject} \]
Multi-counters as an ordered tree

$M_{3,4}$ — complete 4-ary tree of height 2

read 2: move to the smallest leaf in the same level 2 subtree

reject
Multi-counters as an ordered tree

$M_{3,4}$ — complete 4-ary tree of height 2

read 4: move to the smallest leaf in the same level 4 subtree
Multi-counters as an ordered tree

$M_{3,4}$ — complete 4-ary tree of height 2

Question: Do we really need the complete $(n+1)$-ary tree of height $\frac{d}{2}$?
Multi-counters from any ordered tree
Hierarchical Decomposition

D

\[ \leq d/2 \]

\[ \leq n \text{ leaves} \]

\( T_0 \)

\( R_3 \)

\( T_2 \)

\( R_2 \)

\( T_1 \)

\( R_1 \)

\( T_0 \)
**Definition** \( \mu: V \rightarrow \mathbb{I} \): A progress measure for graph \( G \) if for every edge \((v, u) \in E\),

- \( \mu(v)|_{\pi(v)} < \mu(u)|_{\pi(u)} \) if \( \pi(v) \) is odd

- \( \mu(v)|_{\pi(v)} \leq \mu(u)|_{\pi(u)} \) if \( \pi(v) \) is even

\( \leq n \) leaves
**DEFINITION** $\mu : V \rightarrow \mathbb{J}$ is a progress measure for graph $G$ if for every edge $(v, u) \in E$,

- $\mu(v)|_{\pi(v)} < \mu(u)|_{\pi(u)}$ if $\pi(v)$ is odd
- $\mu(v)|_{\pi(v)} \leq \mu(u)|_{\pi(u)}$ if $\pi(v)$ is even
**Definition** \[ \mu : V \rightarrow J \] is a progress measure for graph \( G \) if for every edge \((v, u) \in E:\)

- \[ \mu(v) |_{\pi(v)} < \mu(u) |_{\pi(v)} \] if \( \pi(v) \) is odd
- \[ \mu(v) |_{\pi(v)} \leq \mu(u) |_{\pi(v)} \] if \( \pi(v) \) is even

Diagram:

- Level 1:
  - Node 1
  - Node 4
- Level 2:
  - Node 2
- Level 3:
  - Node 3
- Level 4:
  - Node 4

Arrows indicate the direction of progress measures.
**Definition** \(\mu: V \rightarrow \mathbb{J}\) is a progress measure for graph \(G\) if for every edge \((u,v) \in E\),

- \(\mu(u)_{\pi(u)} < \mu(v)_{\pi(v)}\) if \(\pi(u)\) is odd
- \(\mu(u)_{\pi(u)} \leq \mu(v)_{\pi(v)}\) if \(\pi(u)\) is even
**DEFINITION** \( \mu : V \rightarrow J \) is a progress measure for graph \( G \) if for every edge \((v, u) \in E\),

- \( \mu(v)_{|\pi(v)} < \mu(u)_{|\pi(u)} \) if \( \pi(v) \) is odd

- \( \mu(v)_{|\pi(v)} \leq \mu(u)_{|\pi(u)} \) if \( \pi(v) \) is even

Diagram:
- Node 1 connected to nodes 3 and 4,
- Node 3 connected to node 4,
- Node 4 connected to node 1.

Levels:
- Level 1: Node 1
- Level 2: Node 3, Node 4
- Level 3: Node 1, Node 2
- Level 4: Node 0, Node 1

Values:
- Node 1: 1
- Node 2: \( \mu \)
- Node 3: 0 (red)
- Node 4: 1 (green)
- Node 0: 0

Comparisons:
- Node 3 < Node 4 (red triangle)
- Node 1 = Node 2 (blue triangles)
- Node 1 > Node 0 (red triangle)
A "FUNDAMENTAL THEOREM" OF PARITY GAMES

**THEOREM** [...] Emerson, Jutla 1991; [...]

G has a progress measure iff every cycle in G is even

- height \( d/2 \)
- \( \leq n \) leaves
The multi-counter method

\[ M_{n,d} = \{ (c_{d-1}, c_{d-3}, \ldots, c_3, c_1) : 0 \leq c_p \leq n \} \]

**Fact**  Parity game \( G \) and safety game \( G \triangleright M_{n,d} \) have the same winners.
Multi-counters from any ordered tree
**The multi-counter method**

**Thm.** If $S = M_T$, where $T$ is

- the complete $n$-ary tree of height $\frac{d}{2}$

then Parity game $G$

and safety game $G \models S$

have the same winners
THE MULTI-COUNTER METHOD

**Thm** If $S = M_T$ where $T$ is

- the complete $n$-ary tree of height $\frac{d}{2}$
- the tree of a progress measure $\leq n$ leaves

then Parity game $G$

and safety game $G \triangleright S$

have the same winners
**The Multi-Counter Method**

**Thm**: If \( S = M_T \) where \( T \) is

- the complete \( n \)-ary tree of height \( \frac{d}{2} \)
- the tree of a progress measure \( \leq n \) leaves
- an \((n, \frac{d}{2})\)-universal tree

then Parity game \( G \) and safety game \( G > S \) have the same winners.
**Theorem** [J., Lazić 2017]

There is an \((n, h)\)-universal tree of size \(n \cdot \binom{\lg n + h}{\lg n}\)

- \(\text{poly}(n)\) if \(h = O(\log n)\)
- \(\log \left( \frac{d}{\lg n} \right) + o(1)\) if \(h = \omega(\log n)\)
Universal Trees: A Quasi-Polynomial Upper Bound

[(4,2)]-universal ordered tree

height $h$

\[\{00<0<01<\varepsilon<10<1<11\}\]

the linear order on bit strings induced by $0<\varepsilon<1$

\[\leq \log n \text{ bits in total on every path}\]
If an ordered tree $T$ has height $h$ and $\leq n$ leaves, then there is an order-preserving labelling of $T$ by $\{0,1\}^*$ and $<$ s.t. on every path (from root to leaf) $\leq \log n$ bits are used.
**Lemma**

If an ordered tree $T$ has height $h$ and $\leq n$ leaves, then there is an order-preserving labelling of $T$ by $(\{0,1\}^*, \prec)$ such that on every path (from root to leaf) $\leq \log_2 n$ bits are used.
If an ordered tree $T$ has height $h$ and $\leq n$ leaves, then there is an order-preserving labelling of $T$ by $\{0,1\}^*$, $<$, s.t. on every path (from root to leaf) $\leq \log n$ bits are used.
Lemma

If an ordered tree $T$ has height $h$ and $\leq n$ leaves then there is an order-preserving labelling of $T$ by $(\{0,1\}^*, \prec)$ s.t. on every path (from root to leaf) $\leq \log n$ bits are used.
**Succinct Multi-Counters**

\[ S_{n,h} \overset{d}{=} \left\{ \langle s_h, s_{h-1}, \ldots, s_1 \rangle : s_i \in \{0,1\}^* \text{ and } \sum_{i=1}^{h} |s_i| \leq \lfloor \log n \rfloor \right\} \]

The number of bits sufficient to represent a succinct multi-counter is:

\[ \text{Fact } |S_{n,h}| \leq 2^{\lfloor \log n \rfloor \cdot (1 + \lfloor \log h \rfloor)} = \log h + o(1) \]

For each bit, the bit coordinate of the bit belongs to.
**The size of** $S_{n,h}$

- $|S_{n,h}| \leq 2^\lceil \log n \rceil \cdot \binom{\lceil \log n \rceil + h}{h}$

- $|S_{n,h}| = \begin{cases} 
O(n \cdot \log^h n) & \text{if } h = O(1) \\
O(n^{1+o(1)}) & \text{if } h = o(\log n) \\
\tilde{\Theta}(n^{\log (\delta+1) + \log (e_\delta) + 1}) & \text{if } h = \lceil \delta \cdot \log n \rceil \\
\Theta(n \cdot \log(h \cdot \log n) + 0.45) & \text{if } h = \omega(\log n) 
\end{cases}
$

where $e_\delta = (1 + \frac{1}{\delta})^\delta$
**Theorem** [J., Lazic 2017]

There is an \((n, h)\)-universal tree

of size \(n \cdot \binom{\log n + h}{h} = n^{\log (\frac{h}{\log n}) + o(1)}\)

---

**Theorem** [Czerwinski, Daviaud, Fijalkow, J., Lazic, Pany, 2019]

Every \((n, h)\)-universal tree

is of size at least \(\binom{\log n + h - 1}{h - 1} \geq n^{\log (\frac{h}{\log n}) - 1}\)
**Smallest Universal Trees Are Quasi-Polynomial**

\[ L(n, h) \overset{d}{=} \text{smallest } \# \text{ leaves in an } (n, h) \text{-universal tree} \]

\[
\begin{align*}
L(n, 1) &= n \\
L(1, h) &= 1
\end{align*}
\]

**Upper Bound Recurrence**

\[ L(n, h) \leq L(n, h-1) + 2 \cdot L\left(\frac{n}{2}, h\right) \]

**Lower Bound Recurrence**

\[ L(n, h) \geq L(n, h-1) + L\left(\frac{n}{2}, h\right) \]

**Corollary**

\[ L(n, h) \geq \binom{\log n + h - 1}{h - 1} \]
Solving Parity Games Using Safety Automata

[Bernet, Jamin, Walukiewicz 2003]

The "chained product" game is played on $G \rhd A$

- A reads priority sequences
- A declares Odd the winner if REJECT state is reached

Safety game
**Question** What properties of $A$ make $G$ and $G \triangleright A$ have the same winners?

- If Even plays a positional winning strategy in $G$ then $A$ accepts.
- If Odd plays a positional winning strategy in $G$ then $A$ rejects.
LANGUAGES OF PRIORITY SEQUENCES

\[ \limsup_{\text{Even}}^d \subseteq \{1,2,\ldots,d\}^\omega : \text{won by Even} \]
LANGUAGES OF PRIORITY SEQUENCES

- $\text{LimsupEven}^d \subseteq \{1,2,\ldots,d\}^\omega$: won by Even

- $\text{EvenCycles}_n^d \subseteq \{1,2,\ldots,d\}^\omega$: arising from a game graph (with $6n$ vertices and $\leq d$ priorities) in which all cycles are even
**Definition** A finite (safety) automaton $S$ is an $(n,d)$-separator if $L(S)$ separates $\text{EvenCycles}_n^d$ from $\text{OddCycles}_n^d$. 

$\text{EvenCycles}_n^d$ \hspace{5cm} $\text{OddCycles}_n^d$
**Definition**

A finite (safety) automaton $S$ is a strong $(n,d)$-separator if $L(S)$ separates $\text{EvenCycles}_n^d$ from $\text{LimsupOdd}_n^d$. 

$\text{LimsupEven}_n^d$ $\text{LimsupOdd}_n^d$

$\text{EvenCycles}_n^d$ $\text{OddCycles}_n^d$
**FACT**

If $S$ is an $(n,d)$-separator and the parity game $G$ has $\leq n$ vertices and $\leq d$ priorities, then games $G$ and $G \uparrow S$ have the same winners.
A QUASI-POLYNOMIAL SEPARATOR

**Theorem** [Calude et al. 2017; Bojańczyk, Czerwiński 2019]

There is a strong $(n,d)$-separator of size $n^{\log d + o(1)}$

- **States:** "play statistics" $(P_{\log n}, P_{\log n-1}, \ldots, P_1, P_0) \in \{0,1,2,\ldots,d\}^{\log n+1}$

- **Transitions:** see [Calude et al. 2017] or [Bojańczyk, Czerwiński 2018]

  \[
  \log d \cdot \log n \text{-space TM}
  \]

  \[
  \text{automaton of size } n^{\log d + o(1)}
  \]
**Theorem** [Czerwiński, Daviaud, Fijalkow, J., Lazić, Pany; 2019]

Leaves of every \((n, \frac{d}{2})\)-universal tree are the states of a strong \((n, d)\)-separator.
$\mathcal{M}_u$ REJECTS ALL WORDS IN $\text{Odd}_d$
$M_u$ accepts all words in $\text{Even}_n$.

- If all cycles in $G$ are even
  then there is a progress measure

  \[ \mu : V \rightarrow T \leftarrow U \]

- Let $M_u$ run on $G$:

  Invariant: when $M_u$ visits vertex $v$,
  its state is $\leq \mu(v)$
Theorem [Lehtinen 2018]

There is a parity strong \((n,d)\)-separator with \(\log n\) priorities and of size \(n \cdot d^{\log n} = n^{\log d + O(1)}\).

Corollary

Parity games can be solved in time \((n^{\log d})^{\log n}\).
**Theorem** [Czerwiński, Daviaud, Fijalkow, J., Lazić, Pany 2019]

Leaves of every \((n, \frac{d}{2})\)-universal tree are the states of a strong \((n,d)\)-separator.

---

**Theorem** [Czerwiński, Daviaud, Fijalkow, J., Lazić, Pany 2019]

States in every strong \((n,d)\)-separator include all the leaves in an \((n, \frac{d}{2})\)-universal tree.
Universal trees and separating automata are quasi-polynomial

**Theorem** [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

The sizes of smallest universal trees and of smallest separating automata are quasi-polynomial.

---

Universal trees are separating automata

---

**Theorem** [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

Leaves of every \((n, \frac{d}{2})\)-universal tree are the states of a strong \((n, d)\)-separator.

---

**Theorem** [J., Lazić 2017]

There is an \((n, \frac{d}{2})\)-universal tree of size \(n \left(\frac{\log n + \frac{d}{2}}{\log n}\right) = n^{\log \left(\frac{d}{2} + 0(1)\right)}\).

---

**Theorem** [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

States in every strong \((n, d)\)-separator include all the leaves in an \((n, \frac{d}{2})\)-universal tree.

---

**Theorem** [Czerwiński, Daviaud, Fijalkow, J., Lazić, Parys 2018]

Every \((n, \frac{d}{2})\)-universal tree is of size at least \(\left(\frac{\log n + \frac{d}{2}}{\log n - 1}\right) \geq n^{\log \left(\frac{d}{2}\right) - 2}\).
Hot off/on the press

- [Daviaud, J., Lehtinen 2019]
  "Alternating weak automata from universal trees"

- [Daviaud, J., Thajaswini 20??]
  "The Strahler number of a parity game"

- [Parns 2019]
  "Zielonka’s algorithm in quasi-polynomial time"

- [Lehtinen, Schewe, Wojtczak 2019]
  "Improving the complexity of Parns’ recursive algorithm"

- [J., Monan 20??]
  "A universal algorithm for parity games and model checking"
Fig. 2: The Horton–Strahler rule for the Garonne river network
Open Problems

- Is there a (randomized) poly-time algorithm?
- Design separators that are smaller than strong separators
- Design parity separators that are smaller than safety separators
- Close the $\Omega(n \log n)$ and $n^{o(\log d)}$ gap for the blow-up in alternating parity to weak automata translation
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