A report on NSOP$_1$ theories

Byunghan Kim

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Outline

- A. Chernikov and N. Ramsey, On model theoretic tree properties, J. of Math. Logic (2016).
- I. Kaplan and N. Ramsey, On Kim-independence, To appear in J. of European Math. Society.
- I. Kaplan and N. Ramsey, Transitivity of Kim-independence, Preprint.
- J. Dobrowolski, B. Kim, and N. Ramsey, Independence over arbitrary sets in NSOP_1 theories. Preprint.
- B. Kim, On the number of countable models of a countable NSOP_1 theory without weight ω, Preprint.
- A. Chernikov, B. Kim, and N. Ramsey, Transitivity, lowness and ranks in NSOP_1 theories, Preprint.

We work in a fixed saturated model $\mathcal{M}$ of $T = \mathrm{Th}(\mathcal{M})$ in $\mathcal{L}$.

Definition

- A formula $\varphi(x, a_0)$ *divides over* A if there is an A-indiscernible sequence $\langle a_i |\ i < \omega\rangle$ such that $\{\varphi(x, a_i) | i < \omega\}$ is inconsistent.
- A type $p(x)$ *divides over* A if $p \vdash \varphi(x, a)$ and $\varphi(x, a)$ divides over A.
- A type $p(x)$ *forks over* A if $p \vdash \varphi_0(x, a_0) \vee ... \vee \varphi_n(x, a_n)$ and $\varphi_i(x, a_i)$ divides over A for each $i \leq n$.

Write $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$ if for any $a \in A$, $\mathrm{tp}(a/BC)$ *does not fork over* B.

Let $A \subseteq B \subseteq C$.

- (Extension) If $d \mathop{\smile\!\!\!\!\!\!\mid}_A B$, then there is $d' \equiv_B d$ such that $d' \mathop{\smile\!\!\!\!\!\!\mid}_A C$.
- (Base monotonicity) If $d \mathop{\smile\!\!\!\!\!\!\mid}_A C$ then $d \mathop{\smile\!\!\!\!\!\!\mid}_B C$ holds.

Fact (K.)

The following are equivalent.

- *(Symmetry)* $A \downarrow_B C$ *iff* $C \downarrow_B A$.
- *(Transitivity) For* $A \subseteq B \subseteq C$, *if* $D \downarrow_A B$ *and* $D \downarrow_B C$ *then* $D \downarrow_A C$ *holds.*
- *(Local character) For any set* A *and finite* d, *there is* $A_0 (\subseteq A)$ *of size* $\leq |T|$ *such that* $d \downarrow_{A_0} A$.

Definition

- T is said to be *simple* if one of the above equivalent properties of nonforking holds.
- T is *unstable* if there are $\varphi(x, y)$ and $a_i, b_i \in \mathcal{M}$ $(i < \omega)$ such that $\mathcal{M} \models \varphi(a_i, b_j)$ iff $i < j$.
- T is *stable* if it is not unstable.

Any stable theory is simple.

Definition

$I = \langle a_i \mid i < \omega \rangle$ is said to be a *Morley sequence* over A (or say, in $\mathrm{tp}(a_0/A)$) if I is A-indiscernible, and $a_i \mathop{\smile\!\!\!\!\!\!|}_A a_{<i}$ for all $i < \omega$.

If T is simple then for any $p(x) \in S(A)$, there is a Morley sequence in p.

Kim's Lemma

Assume T is simple(= local character for $\mathop{\smile\!\!\!\!\!\!|}$). Then the following are equivalent.

- *$\varphi(x, a_0)$ divides over A.*
- *For some Morley sequence $\langle a_i \mid i < \omega \rangle$ over A, $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.*
- *For any Morley sequence $\langle a_i \mid i < \omega \rangle$ over A, $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.*

Corollary

If T is simple then the notions of forking and dividing coincide.

Theorem (K., Pillay)

- *If T is simple then $\downarrow$ additionally satisfies the following axiom (The Independence Theorem (or 3-amalgamation) over models): If $c_0 \equiv_M c_1$, $a_0 \downarrow_M a_1$, and $c_i \downarrow_M a_i$ $(i = 0, 1)$, then there is $c \equiv_{Ma_i} c_i$ such that $c \downarrow_M a_0 a_1$.*
- *The five basic axioms together with the above axiom characterize simplicity and forking.*

Stable	Simple	NSOP_1
Infinite set	The random graph	The parametrized equi. rel.s
ACF	Bounded PAC fields	ω-free PAC fields
$V =$ vector sapce	$(V, \langle , \rangle)$ / a finite F	$(V, \langle , \rangle)$ / an infinite F

Definition

We say T has the *tree property* if there exist a formula $\varphi(x, y)$ and tuples c_α with $\alpha \in \omega^{<\omega}$ such that
(i) for all $\beta \in \omega^\omega$, $\{\varphi(x, c_{\beta\lceil n}) \mid n \in \omega\}$ is consistent; and
(ii) for any $\alpha \in \omega^{<\omega}$, $\{\varphi(x, c_{\alpha\frown n}) \mid n \in \omega\}$ is 2-inconsistent.

Proposition (Shelah)

T is simple iff T does not have the tree property.

Shelah introduced the notion of SOP$_1$.

Definition

- We say a formula $\varphi(x,y)$ has *SOP*$_1$ if there is a set of tuples $\{c_\alpha \mid \alpha \in 2^{<\omega}\}$ such that
  (i) for any $\beta \in 2^\omega$, $\{\varphi(x, c_{\beta\restriction m}) \mid m \in \omega\}$ is consistent; and
  (ii) $\{\varphi(x, c_{\alpha\frown\langle 1\rangle}), \varphi(x, c_\gamma)\}$ is inconsistent whenever
  $$\alpha\frown\langle 0\rangle \trianglelefteq \gamma \in 2^{<\omega}.$$
- We say T has SOP$_1$ if for some formula has SOP$_1$.
- We say T is NSOP$_1$ if T does not have SOP$_1$.

SOP$_1$ implies the tree property. So any simple theory is NSOP$_1$.

Theorem (Chernikov, Ramsey)

The following are equivalent.

- *T has SOP_1.*
- *There are a formula $\varphi(x, y)$ and an indiscernible sequence $\langle a_i c_i \mid i < \omega \rangle$ such that $\{\varphi(x, a_i) \mid i < \omega\}$ is consistent, $\{\varphi(x, c_i) \mid i < \omega\}$ is inconsistent, and $a_i \equiv_{(ac)_{<i}} c_i$ for each $i < \omega$.*

Definition/Remark

- Let $M(\prec \mathcal{M})$ be a model. We say a global type $q \in S(\mathcal{M})$ is *M-invariant* if for any b, b' if $b \equiv_M b'$ implies $\varphi(x, b) \in q$ iff $\varphi(x, b') \in q$.
- Fact: Any type $p \in S(M)$ has a global extension which is M-invariant.
- We say $I = \langle a_i \mid i < \omega \rangle$ is a *global Morley sequence over M*, if there is some M-invariant global type q such that $a_i \models q \restriction Ma_{<i}$ for each $i < \omega$. It easily follows that I is a Morley sequence over M (but not the converse).
- We say $\varphi(x, a_0)$ *Kim-divides over M* if $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent for some global Morley sequence $\langle a_i \mid i < \omega \rangle$ over M.
- A type $p(x)$ *Kim-divides* over M if so does a formula implied from it; $p(x)$ *Kim-forks over M* if $p \vdash \varphi_0(x, a_0) \vee \ldots \vee \varphi_n(x, a_n)$ and $\varphi_i(x, a_i)$ Kim-divides over M for each $i \leq n$.
- Write $a \mathop{\smile\!\!\!\!\!\!\,|}^K_M b$ if $\mathrm{tp}(a/Mb)$ does not Kim-divides over M.

Lemma (Kaplan, Ramsey)

(Kim's lemma for $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^K$ *over models) Let* T *be* $NSOP_1$*. Then the following are equivalent.*

- $\varphi(x, a_0)$ *Kim-divides over* M*.*
- $\{\varphi(x, a_i) \mid i < \omega\}$ *is inconsistent for any global Morley sequence* $\langle a_i \mid i < \omega\rangle$ *over* M*.*

Corollary

(T $NSOP_1$*) Kim-forking = Kim-dividing (over models), so* $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^K$ *satisfies extension over models: If* $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^K_M b$ *then for any* c*, there is* $a' \equiv_{Mb} a$ *such that* $a' \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^K_M bc$*.*

Theorem (Kaplan, Ramsey)

The following are equivalent.

- T *is* $NSOP_1$.
- *(Symmetry of* $\mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K$ *over models)* $a \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K_M b$ *iff* $b \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K_M a$.
- *(Transitivity of* $\mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K$ *over models) For models* $M \prec N$, *if* $d \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}_M N$ *and* $d \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}_N b$ *then* $d \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K_M Nb$ *holds.*
- *(3-amalgamation of* $\mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K$ *over models) If* $c_0 \equiv_M c_1$, $a_0 \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K_M a_1$, *and* $c_i \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K_M a_i$ $(i = 0, 1)$, *then there is* $c \equiv_{Ma_i} c_i$ *such that* $c \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K_M a_0 a_1$.

Moreover above two axioms together with strong finite character, existence, and witness axioms characterize $NSOP_1$ *and Kim-dividing over models.*

But in general *base monotonicity* fails to hold in NSOP_1 T:

- For models $M \prec N$, $d \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K_M Nb$ need not imply $d \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K_N b$.
- $a \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K_M bc$ and $b \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K_M c$ need not imply $ab \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\mid$}}^K_M c$.

Proposition (Kaplan, Ramsey)

Assume T is NSOP$_1$. The following are equivalent.

- $\varphi(x, a_0)$ *Kim-divides over* M.
- *For any Morley sequence* $\langle a_i \mid i < \omega \rangle$ *over* M, $\{\varphi(x, a_i) \mid i < \omega\}$ *is inconsistent.*

Nonforking Existence

Any formula over a set does not fork over the set. Equivalently, in any $p(x) \in S(A)$, there is a Morley sequence in p.

For the rest, unless said otherwise. we assume T **is NSOP$_1$ and has nonforking existence**. (Any simple T has nonforking existence. Does any NSOP$_1$ T have this too?)

Even in a simple T, for a set A, a type $p(x) \in S(A)$ need not have a global A-invariant extension.

Definition

$\varphi(x, a_0)$ *Kim-divides over A* if for some Morley sequence $\langle a_i \mid i < \omega \rangle$, $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent. Similalry define for a type to *Kim-divide/fork* over a set.

Write $a \mathop{\smile\!\!\!\!\!\!\mid}^K_A b$ if $\mathrm{tp}(a/Ab)$ does not Kim-divide over A. This notion is coherent with the case when A is a model.

If T is simple then by Kim's lemma, $\mathop{\smile\!\!\!\!\!\!\mid}^K = \mathop{\smile\!\!\!\!\!\!\mid}$.

Definition

- Tuples a, b have the same *Lascar (strong) type* over A (write $a \equiv^L_A b$, or $Lstp(a/A) = Lstp(b/A)$) if there are $a = a_0, a_1, \ldots, a_n = b$ such that for each $i < n$, $a_i a_{i+1}$ begin an indiscernible sequence over A.
- Tuples a, b have the same *KP-type* over A (write $a \equiv^{KP}_A b$) if they are in the same class of every A-type-definable bounded equivalence relation.
- Tuples a, b have the same *strong type* over A (write $a \equiv^s_A b$) if they are in the same class of every A-definable finite equivalence relation.

In general, $a \equiv^L_A b \quad \Rightarrow \quad a \equiv^{KP}_A b \quad \Rightarrow \quad a \equiv^s_A b$.

Theorem (K., Pillay)

Let T be simple.

- *(3-amalgamation for Lascar types) If $a_0 \equiv^L_A a_1$, $c_0 \mathop{\smile\!\!\!\!\!\!\;|}_A c_1$, and $a_i \mathop{\smile\!\!\!\!\!\!\;|}_A c_i$ $(i = 0, 1)$, then there is $a \equiv^L_{Ac_i} a_i$ such that $a \mathop{\smile\!\!\!\!\!\!\;|}_A c_0 c_1$.*
- *T is G-compact. (That is, $a \equiv^L_A b \quad \Leftrightarrow \quad a \equiv^{KP}_A b$.)*

Lemma (Dobrowolski, K., Ramsey)

(Kim's lemma for $\mathop{\smile\hspace{-0.9em}\mid}^K$ over sets) The following are equivalent.

- $\varphi(x, a_0)$ *Kim-divides over a set* A.
- $\{\varphi(x, a_i) \mid i < \omega\}$ *is inconsistent for any Morley sequence* $\langle a_i \mid i < \omega\rangle$ *over* A.

Theorem (Dobrowolski, K., Ramsey)

- *Kim-dividing = Kim-forking (over sets).*
- *(Extension) If* $d \mathop{\smile\hspace{-0.9em}\mid}^K_A B$, *then there is* $d' \equiv_B d$ *such that* $d' \mathop{\smile\hspace{-0.9em}\mid}^K_A C$.
- *(Symmetry)* $a \mathop{\smile\hspace{-0.9em}\mid}^K_A b$ *iff* $b \mathop{\smile\hspace{-0.9em}\mid}^K_A a$.
- *(Type amalgamation for Lascar types w.r.t.* $\mathop{\smile\hspace{-0.9em}\mid}^K$*) If* $a_0 \equiv^L_A a_1$, $c_0 \mathop{\smile\hspace{-0.9em}\mid}^K_A c_1$, *and* $a_i \mathop{\smile\hspace{-0.9em}\mid}^K_A c_i$ $(i = 0, 1)$, *then there is* $a \equiv^L_{Ac_i} a_i$ *such that* $a \mathop{\smile\hspace{-0.9em}\mid}^K_A c_0 c_1$.

Corollary

T is G-compact.

Lachlan's Problem

(T is any countable theory.)

If $1 < I(\omega, T) =$ the no. of countable non-isomorphic models of $T < \omega$ then T has the strict order property.

Recall that T has the *strict order property* if there is a definable ordering with an infinite chain, equivalently there are $\varphi(x, y)$ and a_i $(i < \omega)$ such that

$$\models \varphi(x, a_j) \rightarrow \varphi(x, a_i) \text{ iff } i < j < \omega.$$

Definition

- T is said to be *supersimple* if for any finite d, and a set A there is $A_0(\subseteq A)$ of size $\leq |T|$ such that $d \downarrow_{A_0} A$.
- T is *superstable* if T is stable and supersimple.

Theorem (Lachlan, 70s)

If countable T is superstable then $I(\omega, T)$ is either 1 or infinite.

Theorem (K., 90s)

If T is countable supersimple then $I(\omega, T)$ is either 1 or infinite.

Theorem (Pillay)

If T is countable stable and $1 < I(\omega, T) < \omega$ then there is a finite tuple whose own preweight is ω.

We return back to assume T is NSOP_1 with existence.

Definition

We say a tuple c (or its type) has *own preweight* ω if there are $b_i \equiv c$ $(i < \omega)$ such that $c \not\mathop{\smile\hskip-0.9em ^|}\ ^K b_i$, and $b_i \mathop{\smile\hskip-0.9em ^|}\ ^K b_{<i}$ for all $i < \omega$.

Theorem (K.)

If T is countable and $1 < I(\omega, T) < \omega$ then there is a finite tuple whose own preweight is ω.

Theorem (Kaplan, Ramsey)

(Transitivity) Let $A \subseteq B \subseteq C$. If $d \mathop{\smile\!\!\!\!\!\!|}^K_A B$ and $d \mathop{\smile\!\!\!\!\!\!|}^K_B C$ then $d \mathop{\smile\!\!\!\!\!\!|}^K_A C$.

Definition

An A-indiscernible sequence $\langle a_i \mid i < \omega \rangle$ is said to be $\mathop{\smile\!\!\!\!\!\!|}^K$-*Morley* over A if $a_i \mathop{\smile\!\!\!\!\!\!|}^K_A a_{<i}$ for all $i < \omega$.

Corollary

The following are equivalent.

- $\varphi(x, a_0)$ *Kim-divides over A.*
- *For any $\mathop{\smile\!\!\!\!\!\!|}^K$-Morley $\langle a_i \mid i < \omega \rangle$ over A, $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.*
- *For some $\mathop{\smile\!\!\!\!\!\!|}^K$-Morley $\langle a_i \mid i < \omega \rangle$ over A, $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.*

Definition

We say T is *low* if for any $\mathcal{L}$-formula $\varphi(x, y)$, there is $k \geq 2$ such that whenever $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent for some indiscernible $\langle a_i \mid i < \omega \rangle$, then it is k-inconsistent.

Hence if T is low then for any $\varphi(x, y)$, there is a type $q(y)$ over $\emptyset$ such that

$$q(b) \text{ holds if and only if } \varphi(x, b) \text{ divides over } \emptyset.$$

Theorem (Buechler)

If T is low simple then strong types are Lascar types.

Theorem (Chernikov, K., Ramsey)

If T is low then strong types are Lascar types.

(T is any theory.)

Fact

- *T is simple iff there do not exist finite d and increasing sets $\langle A_i \mid i < |T|^+ \rangle$ such that $d \not\mathop{\smile\!\!\!\!\!\!|}_{A_i} A_{i+1}$ for each $i < |T|^+$.*
- *T is supersimple iff there do not exist finite d and increasing sets $\langle A_i \mid i < \omega \rangle$ such that $d \not\mathop{\smile\!\!\!\!\!\!|}_{A_i} A_{i+1}$ for each $i < \omega$.*

Fact (Kaplan, Ramsey, Shelah)

T is NSOP$_1$ iff there do not exist finite d and a continuous increasing sequence $\langle M_i \mid i < |T|^+ \rangle$ of $|T|$-sized models such that $d \not\mathop{\smile\!\!\!\!\!\!|}^K_{M_i} M_{i+1}$ for each $i < |T|^+$.

Fact (Casanovas, K.)

T is supersimple iff there do not exist finite d and increasing sets $\langle A_i \mid i < \omega \rangle$ such that $d \not\mathop{\smile\!\!\!\!\!\!|}^K_{A_i} A_{i+1}$ for each $i < \omega$.

Definition

- A binary graph $V = (V, E)$ with $|V| \geq 2$ is said to be *nice* if there are no triangles and squares; and for any two distinct $a, b \in V$ there is $c \in V$ such that $E(a, c)$ but $\neg E(b, c)$.
- Fix an odd prime p. For a nice graph V, let $F(V)$ be the free nilpotent group of class 2 and exponent p generated by the vertices of V. Assume that V is enumerated as $V = \{a_i \mid i < \kappa\}$. Then the *Mekler group* of V, written $G(V)$, is defined as

$$G(V) := F(V)/\langle [a_i, a_j] \mid i < j \text{ and } E(a_i, a_j)\rangle.$$

Theorem (Ahn)

Let $V = (V, E)$ be a nice graph. Then $\mathrm{Th}(V)$ *is NSOP$_1$ iff so is* $\mathrm{Th}(G(V))$.

- Does nonforking existence hold in any NSOP_1 T?

Shelah indeed introduced the notions of SOP_n for $n \geq 1$.

Definition

- T has SOP_n ($n \geq 3$) if there is a formula $\varphi(x, y)$ ($|x| = |y|$) defining a directed graph having an infinite chain but no cycle of length $\leq n$.
- T has SOP_2 if there are a formula $\varphi(x, y)$ and tuples c_α ($\alpha \in 2^{<\omega}$) such that, for each $\beta \in 2^\omega$, $\{\varphi(x, c_{\beta\lceil i} \mid i \in \omega\}$ is consistent; while for any incomparable $\alpha, \gamma \in 2^{<\omega}$, $\varphi(x, a_\alpha) \wedge \varphi(x, a_\gamma)$ is inconsistent.

Definition

- T has TP_1 if there are a formula $\varphi(x, y)$ and tuples c_α ($\alpha \in \omega^{<\omega}$) such that, for each $\beta \in \omega^\omega$, $\{\varphi(x, c_{\beta\lceil i} \mid i \in \omega\}$ is consistent; while for any incomparable $\alpha, \gamma \in \omega^{<\omega}$, $\varphi(x, a_\alpha) \wedge \varphi(x, a_\gamma)$ is inconsistent.
- T has TP_2 if there are a formula $\varphi(x, y)$ and tuples c_{ij} ($i, j < \omega$) such that, for each $i \in \omega$, $\{\varphi(x, c_{ij} \mid j \in \omega\}$ is 2-inconsistent; while for any $f : \omega \to \omega$, $\{\varphi(x, a_{if(i)}) \mid i < \omega\}$ is consistent.

Fact

- *If T has the strict order property then it has SOP_n for each $n \geq 1$.*
- *If T has SOP_{n+1} then it has SOP_n for each $n \geq 1$. But the converse does not hold if $n \geq 3$.*
- *T has the tree property iff T has TP_1 or TP_2.*
- *T has TP_1 iff it has SOP_2.*
- *If T has SOP_1 then T has the tree property.*

- Are the notions of SOP_1 and SOP_2 equivalent?

- Do canonical bases for Lascar types exist in any $NSOP_1$ T?
- Develop a theory of groups in $NSOP_1$ theories.