Outline

- Introduction: Quasi-isometry between colored metric spaces
- Structure of $\leq_{QI}$
  - Lemmas: small cross-over, decomposition, reduction
  - Structure theorems: infinite chain, infinite antichain, density, etc.
- Problems on $\leq_{QI}$
  - Büchi automata and large scale geometries
  - Complexity of the quasi-isometry problem
  - Asymptotic cones
This talk is based on the following papers:


The slide is available at my webpage

- http://group-mmm.org/~toru/
Quasi-isometries

Let \((M_1, d_1)\) and \((M_2, d_2)\) be metric spaces.

**Definition**

A map \(f : M_1 \to M_2\) is an \((A, B, C)\)-quasi-isometry, where \(A \geq 1, B \geq 0\) and \(C \geq 0\), if for all \(x, y \in M_1\) we have

\[
\frac{1}{A} \cdot d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A \cdot d_1(x, y) + B,
\]

and for all \(y \in M_2\) there is an \(x \in M_1\) such that \(d_2(y, f(x)) \leq C\).

When \(B = 0\), the mapping is bi-Lipschitz. Thus, a quasi-isometry is a bi-Lipschitz map with a distortion.
**Definition**

A map $f : M_1 \rightarrow M_2$ is an $(A, B, C)$–quasi-isometry, where $A \geq 1$, $B \geq 0$ and $C \geq 0$, if for all $x, y \in M_1$ we have

$$
\frac{1}{A} \cdot d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A \cdot d_1(x, y) + B,
$$

and for all $y \in M_2$ there is an $x \in M_1$ such that $d_2(y, f(x)) \leq C$.

**Example**

$\mathbb{R}$ and $\mathbb{Z}$ are quasi-isometric.

- The function $f(n) = n$ is a $(1, 0, 1)$–quasi-isometry from $\mathbb{Z}$ to $\mathbb{R}$.
- The function $g(x) = \lceil x \rceil$ is a $(1, 1, 0)$–quasi-isometry from $\mathbb{R}$ to $\mathbb{Z}$.
Examples

**Definition**

A map \( f : M_1 \to M_2 \) is an \((A, B, C)\)-quasi-isometry, where \( A \geq 1 \), \( B \geq 0 \) and \( C \geq 0 \), if for all \( x, y \in M_1 \) we have

\[
\frac{1}{A} \cdot d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A \cdot d_1(x, y) + B,
\]

and for all \( y \in M_2 \) there is an \( x \in M_1 \) such that \( d_2(y, f(x)) \leq C \).

**Example**

Let \( G \) be a finitely generated group, and \( S \) and \( S' \) be its generators. Then the Cayley graphs of \( G \) based on \( S \) and \( S' \) are quasi-isometric.

Proof sketch: if \( |g|_S = n \), then \( |g|_{S'} \leq Mn \), where \( M = \max_{s \in S} |s|_{S'} \). Thus the identity map on \( G \) is a quasi-isometry.
Why do we need quasi-isometries

The notion has been proposed by Gromov for the study of geometric group theory. Studying quasi-isometry (QI) invariants of groups turned out to be crucial in solving many important problems. Hence, finding QI-invariants is an important theme in geometric group theory. Here are examples of QI-invariants:

1. Virtually nilpotent,
2. Virtually free,
3. Hyperbolic,
4. Having polynomial growth rate,
5. Finite presentability,
6. Having decidable word problem,
7. Asymptotic cones, etc.
A coloured metric space is a tuple $\mathcal{M} = (M; d, C)$, where $(M, d)$ is the metric space, and $C$ is a colour function $C : M \to \Sigma$. If $\sigma = C(m)$ then $m$ has colour $\sigma$.

**Example**

Consider $\Sigma^\omega$, the set of infinite strings over $\Sigma$. Each $\alpha \in \Sigma^\omega$ is a coloured metric space.

**Definition**

Let $\mathcal{M}_1 = (M_1; d_1, C_1)$ and $\mathcal{M}_2 = (M_2; d_2, C_2)$ be coloured metric spaces. A colour preserving $(A, B, C)$–quasi-isometry from $(M_1; d_1)$ into $(M_2; d_2)$ is a $(A, B, C)$–quasi-isometry from $\mathcal{M}_1$ into $\mathcal{M}_2$.

If there exists such a function from $\mathcal{M}_1$ to $\mathcal{M}_2$, then we write $\mathcal{M}_1 \leq_{QI} \mathcal{M}_2$. 
The relation $\leq_{QI}$

Example

$0^\omega \leq_{QI} (01)^\omega$ holds. **The converse does not hold.**

- Define a function $f : 0^\omega \rightarrow (01)^\omega$ by $f(2^n) = f(2n+1) = 2n$.
- There is no colour-preserving function from $(01)^\omega$ to $0^\omega$.

Example

$01001\ldots 0^n1\ldots \leq_{QI} (01)^\omega$ holds. **The converse does not hold.**
Large scale geometries

Definition

The equivalence classes of $\sim_{QI}$ are the quasi-isometry types or the large scale geometries of $\alpha$. Set $\Sigma_{QI} = \Sigma^\omega / \sim_{QI}$. Denote by $[\alpha]$ the large scale geometry of $\alpha$.

Example

The QI type $[(01)^\omega]$ is the set of all binary strings such that, for some constant $M$, any of its subsequence of the length $M$ contains 0 and 1.

From now on, every coloured metric space that appear in the talk is an infinite string, which is denoted by $\alpha, \beta, \gamma, ...$
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Lemma (Small Cross Over Lemma)

For any given quasi-isometry constants $(A, B, C)$ there are constants $D \leq 0$ and $D' \leq 0$ such that for all quasi-isometry maps $g : \alpha \rightarrow \beta$ we have the following:

1. For all $n, m \in \omega$ if $n < m$ and $g(m) < g(n)$ we have $g(m) - g(n) \geq D$.

2. For all $n, m \in \omega$ if $n < m$ and $g(m) < g(n)$ then $n - m \geq D'$.

Proof idea:
Decomposition Lemma

Lemma (Decomposition Lemma)

There exists a procedure that given \((A, B, C)\)–quasi-isometry \(f : \alpha \to \beta\) produces a decomposition of \(f\) into quasi-isometries \(\alpha \xrightarrow{f_1} \gamma_1 \xrightarrow{f_2} \gamma_2 \xrightarrow{f_3} \beta\) such that each of the following holds:

1. \(f_1\) is a bijection, \(f_2\) is a monotonic injection, and \(f_3\) is a monotonic surjection.
2. \(f_1\) is a monotonic injection, \(f_2\) is a bijection, and \(f_3\) is a monotonic surjection.
3. \(f_1\) is a bijection, \(f_2\) is a monotonic surjection, and \(f_3\) is a monotonic injection.
Proof: decomposition into injection and mono surjection
Proof: injection $\rightarrow$ mono injection and bijection
**Definition**

Say \( \alpha \) is component-wise reducible to \( \beta \), written \( \alpha \leq_{CR} \beta \), if we can partition \( \alpha \) and \( \beta \) as

\[
\alpha = u_1 u_2 \ldots \text{ and } \beta = v_1 v_2 \ldots
\]

such that \( Cl(u_i) \subseteq Cl(v_i) \) for all \( i \) and \( |u_j|, |v_j| \) are uniformly bounded by a constant \( C \). Call these presentations of \( \alpha \) and \( \beta \) witnessing partitions and intervals \( u_i \) and \( v_i \) partitioning intervals.

**Theorem**

\( \alpha \leq_{QI} \beta \) implies \( \alpha \leq_{CR} \beta \).
If the QI map is monotonic, then the proof is easy. It is not in the non-monotonic case. We use a refined version of decomposition theorem and show a transitivity-like lemma. A function of the following form is called an *atomic crossing map*:
Proof idea

**Lemma**

Any bijective quasi-isometry can be decomposed into finite number of atomic crossing maps, each of which are also quasi-isometry.

**Lemma**

Suppose \( \alpha \leq_{QI} \beta \) via an atomic crossing map \( f : \alpha \rightarrow \beta \) and \( \beta \leq_{CR} \gamma \). Then \( \alpha \leq_{CR} \gamma \).

(\( \alpha \leq_{QI} \beta \) implies \( \alpha \leq_{CR} \beta \).) Decompose the QI map into \( \alpha \xrightarrow{f_1} \gamma \xrightarrow{f_2} \beta \), where \( f_1 \) is bijective and \( f_2 \) is monotonic. Then apply the lemma above iteratively.
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From now on we assume $\Sigma = \{0, 1\}$.

For $\alpha = 0^{n_0}1^{m_0}0^{n_1}1^{m_1} \ldots \in \{0, 1\}^\omega (n_i, m_i \geq 1)$, we call $0^{n_i}$ and $1^{m_i}$ the 0-blocks and 1-blocks, respectively.

An infinite succession of $\sigma \in \Sigma$ is also called a $\sigma$-block.
We split the set $\Sigma_{QI}^\omega$ into four subsets:

- $\mathcal{X}(0) = \{ [\alpha] \mid \text{in } \alpha \text{ all the lengths of 0-blocks are universally bounded} \}$,
- $\mathcal{X}(1) = \{ [\alpha] \mid \text{in } \alpha \text{ the lengths of all 1-blocks are universally bounded} \}$,
- $\mathcal{X}(u) = \{ [\alpha] \mid \text{in } \alpha \text{ the lengths of both 0-blocks and 1-blocks are unbounded} \}$,
- $\mathcal{X}(b) = \{ [\alpha] \mid \text{in } \alpha \text{ the lengths of both 0-blocks and 1-blocks are universally bounded} \}$.

**Theorem**

The sets $\mathcal{X}(0)$, $\mathcal{X}(1)$, $\mathcal{X}(u)$, $\mathcal{X}(b)$ have the following properties:

1. The sets $\mathcal{X}(0)$ and $\mathcal{X}(1)$ are filters.
2. The set $\mathcal{X}(u)$ is an ideal.
3. The set $\mathcal{X}(b)$ is the singleton $\{ [(01)\omega] \}$. 
The set $\mathcal{X}(b)$ is the singleton $\{(01)^{\omega}\}$, and is the greatest element.

The sets $\mathcal{X}(0)$ and $\mathcal{X}(1)$ are filters.

The set $\mathcal{X}(u)$ is an ideal.

$[0^{\omega}]$ and $[1^{\omega}]$ are minimal.
Structure theorems

- $X(0)$, $X(1)$ and $X(u)$ contain chains $(\alpha_n)_{n \in \mathbb{Z}}$ of the type of integers, that is
  \[ \forall n \in \mathbb{Z} [\alpha_n <_{QI} \alpha_{n+1}] \]

Proof:

\[ \alpha_1 = 0101001001 \ldots 0^{2^n} 10^{2^n} 1 \ldots \]
\[ \alpha_0 = 01001 \ldots 0^{2^n} 1 \ldots \]
\[ \alpha_{-1} = 0100001 \ldots 0^{4^n} 1 \ldots \]
\( \mathcal{X}(0), \mathcal{X}(1) \) and \( \mathcal{X}(u) \) have countable antichains.

Proof:

\[ \beta_n = 010^{2n}1^{2n}0^{3n}1^{3n}...0^{kn}1^{kn}... \]
\( \Sigma_{QI}^\omega \) possesses infinitely many minimal elements.

Proof. For any unbounded nondecreasing sequence \( \{a_n\}_{n \in \omega} \), the following sequence is minimal:

\[
\alpha = 0^{a_0}1^{a_1}0^{a_2}1^{a_3}...0^{a_{2k}}1^{a_{2k+1}}...
\]

Problem: Are there uncountably many minimal elements?
Let $\alpha, \beta \in \Sigma^\omega$ be given. Assume $\alpha <_{QI} \beta$, and every letter in $\alpha$ or $\beta$ occurs in both $\alpha$ and $\beta$ infinitely many often. Then there exists $\gamma \in \Sigma^\omega$ such that $\alpha <_{QI} \gamma <_{QI} \beta$.

Moreover, there are infinitely many $\gamma$’s that satisfy this inequality, and not quasi-isometric each other.
Some naive definitions turn out to be not well-defined (i.e. there are $\alpha \sim_{QI} \alpha'$ and $\beta \sim_{QI} \beta'$ such that $[\alpha \land \beta] \neq [\alpha' \land \beta']$).

- $\alpha \land \beta = \alpha(0)\beta(0)\alpha(1)\beta(1) \ldots$
- $\alpha \land \beta = \alpha \text{ XOR } \beta$

Theorem (Stephan+, personal communication)

There are strings $\alpha$ and $\beta$ for which no least upper bound exist.
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**Definition**

An atlas is a set of quasi-isometry types. In particular, the atlas defined by the language $L$ is the set $[L] = \{ [\alpha] \mid \alpha \in L \}$, where $[\alpha]$ is the quasi-isometry type of $\alpha$.

**Definition**

A Büchi automaton $\mathcal{M}$ is a quadruple $(S, \iota, \Delta, F)$, where $S$ is a finite set of states, $\iota \in S$ is the initial state, $\Delta \subseteq S \times \Sigma \times S$ is the transition table, and $F \subseteq S$ is the set of accepting states.
Theorem

Any atlas \([L]\) defined by a Büchi recognisable language \(L\) is a union from the following list of atlases:

- \(\{(01)^\omega\}\), \(\{1^\omega\}\), \(\{0^\omega\}\), \(\{01^\omega\}\), \(\{10^\omega\}\),
- \(\Sigma_{QI}^\omega \setminus \{[0^\omega], [1^\omega], [10^\omega], [01^\omega]\}\),
- \(\mathcal{X}(0) \setminus \{[1^\omega], [01^\omega]\}\),
- \(\mathcal{X}(1) \setminus \{[0^\omega], [10^\omega]\}\).
Geometries of strings accepted by $\mathcal{M}$

**Theorem**

Any atlas $[L]$ defined by a Büchi recognisable language $L$ is a union from the following list of atlases:

- $\{(01)^\omega\}$, $\{1^\omega\}$, $\{0^\omega\}$, $\{01^\omega\}$, $\{10^\omega\}$,
- $\sum_{QI}^\omega \setminus \{[0^\omega], [1^\omega], [10^\omega], [01^\omega]\}$,
- $\mathcal{X}(0) \setminus \{[1^\omega], [01^\omega]\}$,
- $\mathcal{X}(1) \setminus \{[0^\omega], [10^\omega]\}$.

31 / 44
Theorem

Any atlas \([L]\) defined by a Büchi recognisable language \(L\) is a union from the following list of atlases:

- \([\{(01)\omega\}]\), \([\{1\omega\}]\), \([\{0\omega\}]\), \([\{01\omega\}]\), \([\{10\omega\}]\),
- \(\sum_{QI}^\omega \setminus \{[0\omega], [1\omega], [10\omega], [01\omega]\}\),
- \(\mathcal{X}(0) \setminus \{[1\omega], [01\omega]\}\),
- \(\mathcal{X}(1) \setminus \{[0\omega], [10\omega]\}\).
Theorem

Any atlas $[L]$ defined by a Büchi recognisable language $L$ is a union from the following list of atlases:

- $\{(01)^\omega\}$, $\{1^\omega\}$, $\{0^\omega\}$, $\{01^\omega\}$, $\{10^\omega\}$,
- $\Sigma^\omega_{QI} \setminus \{[0^\omega], [1^\omega], [10^\omega], [01^\omega]\}$,
- $\mathcal{X}(0) \setminus \{[1^\omega], [01^\omega]\}$,
- $\mathcal{X}(1) \setminus \{[0^\omega], [10^\omega]\}$.
Call a loop of a Büchi automaton a $0$-loop if only 0 is read through the loop. Define $1$-loops and $01$-loops in a similar way. Then all Büchi automata are categorized by the following features:

- if it has a 0-loop, 1-loop, and 01-loop or not; and
- how these loops are connected.

For example,

- if it has 0-loop and 1-loop, the initial state is in 0-loop and can move from one loop to another, then the automaton accepts $\Sigma_{Q_I}^\omega \setminus [1^\omega]$.
- If it has 0-loop and 01-loop, the initial state is in 0-loop and can move from one loop to another, then the automaton accepts $\mathcal{X}(1)$. 
Corollary

There exists an algorithm that, given Büchi automata $A$ and $B$, decides if the atlases $[L(A)]$ and $[L(B)]$ coincide. Furthermore, the algorithm runs in linear time on the size of the input automata.

In contrast, the problem of deciding whether two given Büchi automata represent the same language is PSPACE-complete.
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Problem and our result

The quasi-isometry problem consists of determining if given two strings $\alpha$ and $\beta$ are quasi-isometric. Formally, the quasi-isometry problem (over the alphabet $\Sigma$) is identified as the set:

$$QIP = \{(\alpha, \beta) \mid \alpha, \beta \in \Sigma^\omega \& [\alpha] = [\beta]\}.$$

Theorem

The following statements are true:

1. Given quasi-isometric strings $\alpha$ and $\beta$, there exists a quasi-isometry between $\alpha$ and $\beta$ computable in the halting set relative to $\alpha$ and $\beta$.

2. The quasi-isometry problem between computable strings, that is the following set $QIP = \{(\alpha, \beta) \mid \alpha, \beta \in \Sigma^\omega, [\alpha] = [\beta], \alpha \text{ and } \beta \text{ are computable}\}$ is a complete $\Sigma^0_2$-set.

Problem

Given quasi-isometric strings $\alpha$ and $\beta$, does there exist a computable quasi-isometry between them?
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Let $\alpha \in \Sigma^\omega$ be a coloured metric space.

- We call $s : \omega \to \omega$ a scaling factor if it is strictly monotonic and $s(0) = 1$.
- Let $d_n(i, j) = |i - j|/s(n)$.

We define the following sequence of metric spaces:

$$X_{0,\alpha} = (\alpha, d_0), \ X_{1,\alpha} = (\alpha, d_1), \ldots, X_{n,\alpha} = (\alpha, d_n), \ldots$$

We want to define a “limit” of this sequence in a formal way to treat the large scale geometry of $\alpha$. We adopt the notion of asymptotic cone to do that.
Let $\mathcal{F} \subset P(\omega)$ be a non-principal ultrafilter.

- Let $a = (a_n)_{n \geq 0}$ be a sequence, where $a_n \in X_{n,\alpha}$.
- $a$ is $\mathcal{F}$-bounded if $\{n \mid d_n(0, a_n) < L\} \in \mathcal{F}$ for some $L$.
- Let $\mathcal{B}(\mathcal{F}, s)$ be the set of all bounded sequences $a = (a_n)_{n \geq 0}$.
- $a, b \in \mathcal{B}(\mathcal{F}, s)$ is said to be $\mathcal{F}$-equivalent ($a \sim_\mathcal{F} c$) if

$$\forall \epsilon > 0 \{n \mid d_n(a_n, b_n) \leq \epsilon\} \in \mathcal{F}.$$
Definition

For given sequence $\alpha$, scaling function $s$ and ultrafilter $\mathcal{F}$, the asymptotic cone of $\alpha$, written $\text{Cone}(\alpha, \mathcal{F}, s)$, with respect to the scaling function $s(n)$ and the ultra-filter $\mathcal{F}$ is the factor set

$$
\mathcal{B}(\mathcal{F}, s)/ \sim_{\mathcal{F}}
$$

equipped with the following metric $D$ and colour $C$:

1. $D(a, b) = r$ if and only if for every $\epsilon$ the set 
   $$
   \{n \mid r - \epsilon \leq d_n(a_n, b_n) \leq r + \epsilon \}
   $$
   belongs to $\mathcal{F}$.

2. $C(a) = \sigma$ if and only if the set 
   $$
   \{n \mid a_n \text{ has colour } \sigma \}
   $$
   belongs to $\mathcal{F}$. 

The following theorems are coloured variant of a known results in geometric group theory, which says the set of all asymptotic cones modulo quasi-isometry is "simpler" than $\Sigma_{QI}^\omega$.

**Theorem**

If strings $\alpha$ and $\beta$ are quasi-isometric then the following holds for the asymptotic cones $\text{Cone}(\alpha, \mathcal{F}, s)$ and $\text{Cone}(\beta, \mathcal{F}, s)$.

1. They are bi-Lipschitz equivalent; i.e. they are quasi-isometric with the additive constant $B = 0$.

2. The bi-Lipschitz map above can be taken as a order preserving map.

**Theorem**

There are two non-quasi-isometric strings $\alpha, \beta \in \{0, 1\}^\omega$, a scale factor $s(n)$, and filter $\mathcal{F}$ such that the cones $\text{Cone}(\alpha, \mathcal{F}, s)$ and $\text{Cone}(\beta, \mathcal{F}, s)$ coincide.
Theorem

If $\alpha$ is Martin-Löf random, then for all computable scaling factors $s$ and ultra-filters $\mathcal{F}$, the asymptotic cone $\text{Cone}(\alpha, \mathcal{F}, s)$ coincides with the space $(\mathcal{R}_{\geq 0}; d, C)$, where all reals have all colours from alphabet $\Sigma$. 
Future work

- Open problems
  - Cardinality of the set of minimal elements
  - Existence of computable QI-map for computable sequences
  - There are some more...

- Degree theory for $\leq_{QI}$ (ongoing w/ F. Stephan, S. Jain)