Large Scale Geometries of Infinite Strings

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Outline

- Introduction: Quasi-isometry between colored metric spaces
- Structure of $\leq_{QI}$
  - Lemmas: small cross-over, decomposition, reduction
  - Structure theorems: infinite chain, infinite antichain, density, etc.
- Problems on $\leq_{QI}$
  - Büchi automata and large scale geometries
  - Complexity of the quasi-isometry problem
  - Asymptotic cones

This talk is based on the following papers:

- Bakh Khoussainov, Toru Takisaka: Large Scale Geometries of Infinite Strings. Proc. LICS 2017.
- Bakh Khoussainov, Toru Takisaka: Infinite Strings and Their Large Scale Properties. Submitted.

The slide is available at my webpage

- http://group-mmm.org/˜toru/

Let (M_1, d_1) and (M_2, d_2) be metric spaces.

Definition

*A map $f : M_1 \to M_2$ is an $(A, B, C)-$***quasi-isometry***, where $A \geq 1$, $B \geq 0$ and $C \geq 0$, if for all $x, y \in M_1$ we have*

$$(1/A) \cdot d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A \cdot d_1(x, y) + B,$$

and for all $y \in M_2$ there is an $x \in M_1$ such that $d_2(y, f(x)) \leq C$.

When $B = 0$, the mapping is bi-Lipshitz. Thus, a quasi-isometry is a bi-Lipschitz map with a distortion.

Examples

Definition

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and for all $y \in M_2$ there is an $x \in M_1$ such that $d_2(y, f(x)) \leq C$.

Example

$\mathbb{R}$ and $\mathbb{Z}$ are quasi-isometric.

- The function $f(n) = n$ is a $(1, 0, 1)-$quasi-isometry from $\mathbb{Z}$ to $\mathbb{R}$.
- The function $g(x) = \lceil x \rceil$ is a $(1, 1, 0)-$quasi-isometry from $\mathbb{R}$ to $\mathbb{Z}$.

Definition

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and for all $y \in M_2$ *there is an* $x \in M_1$ *such that* $d_2(y, f(x)) \leq C$.

Example

Let G be a finitely generated group, and S and S' be its generators. Then the Cayley graphs of G based on S and S' are quasi-isometric.

Proof sketch: if $|g|_S = n$, then $|g|_{S'} \leq Mn$, where $M = \max_{s \in S} |s|_{S'}$. Thus the identity map on G is a quasi-isometry.

Why do we need quasi-isometries

The notion has been proposed by Gromov for the study of geometric group theory.
Studying quasi-isometry (QI) invariants of groups turned out to be crucial in solving many important problems. Hence, finding QI-invariants is an important theme in geometric group theory. Here are examples of QI-invariants:

1. virtually nilpotent,
2. virtually free,
3. hyperbolic,
4. having polynomial growth rate,
5. Finite presentability,
6. Having decidable word problem,
7. Asymptotic cones, etc.

Infinite strings as coloured metric spaces

A **coloured metric space** is a tuple $\mathcal{M} = (M; d, C)$, where (M, d) is the metric space, and C is a colour function $C : M \to \Sigma$. If $\sigma = C(m)$ then m *has colour* σ.

Example

Consdier Σ^ω, the set of infinite strings over Σ. Each $\alpha \in \Sigma^\omega$ is a coloured metric space.

Definition

*Let $\mathcal{M}_1 = (M_1; d_1, C_1)$ and $\mathcal{M}_2 = (M_2; d_2, C_2)$ be coloured metric spaces. A colour preserving $(A, B, C)-$quasi-isometry from $(M_1; d_1)$ into $(M_2; d_2)$ is a $(A, B, C)-$***quasi-isometry** *from $\mathcal{M}_1$ into $\mathcal{M}_2$.*

If there exists such a function from $\mathcal{M}_1$ to $\mathcal{M}_2$, then we write $\mathcal{M}_1 \leq_{QI} \mathcal{M}_2$.

The relation $\leq_{QI}$

Example

$0^\omega \leq_{QI} (01)^\omega$ holds. **The converse does not hold.**

- Define a function $f : 0^\omega \to (01)^\omega$ by $f(2n) = f(2n+1) = 2n$.
- There is no colour-preserving function from $(01)^\omega$ to 0^ω.

Example

$01001\ldots0^n1\ldots \leq_{QI} (01)^\omega$ holds. **The converse does not hold.**

Large scale geometries

Definition

The equivalence classes of $\sim_{QI}$ are the **quasi-isometry types** *or* **the large scale geometries of** α. *Set $\Sigma^{\omega}_{QI} = \Sigma^{\omega} / \sim_{QI}$. Denote by $[\alpha]$* **the large scale geometry of** α.

Example

The QI type $[(01)^{\omega}]$ is the set of all binary strings such that, for some constant M, any of its subsequence of the length M contains 0 and 1.

From now on, every coloured metric space that appear in the talk is an infinite string, which is denoted by $\alpha, \beta, \gamma, ...$

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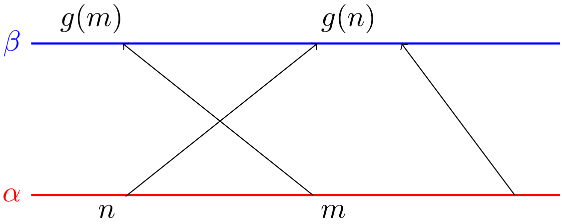
Small Cross-Over Lemma

Lemma (Small Cross Over Lemma)

For any given quasi-isometry constants (A, B, C) *there are constants* $D \leq 0$ *and* $D' \leq 0$ *such that for all quasi-isometry maps* $g : \alpha \to \beta$ *we have the following:*

1. *For all* $n, m \in \omega$ *if* $n < m$ *and* $g(m) < g(n)$ *we have* $g(m) - g(n) \geq D$.
2. *For all* $n, m \in \omega$ *if* $n < m$ *and* $g(m) < g(n)$ *then* $n - m \geq D'$.

Proof idea:

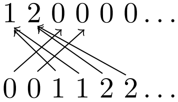


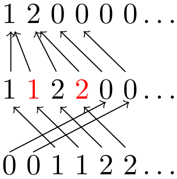
Lemma (Decomposition Lemma)

There exists a procedure that given $(A, B, C)-$quasi-isometry $f : \alpha \to \beta$ produces a decompositon of f into quasi-isometries $\alpha \xrightarrow{f_1} \gamma_1 \xrightarrow{f_2} \gamma_2 \xrightarrow{f_3} \beta$ such that each of the following holds:

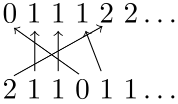
1. *f_1 is a bijection, f_2 is a monotonic injection, and f_3 is a monotonic surjection.*
2. *f_1 is a monotonic injection, f_2 is a bijection, and f_3 is a monotonic surjection.*
3. *f_1 is a bijection, f_2 is a monotonic surjection, and f_3 is a monotonic injection.*

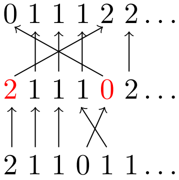
Proof: decomposition into injection and mono surjection





Proof: injection → mono injection and bijection





Componentwise reducibility

Definition

Say α *is* **component-wise reducible** *to* β*, written* $\alpha \leq_{CR} \beta$*, if we can partition* α *and* β *as*

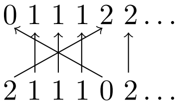
$$\alpha = u_1 u_2 \ldots \text{ and } \beta = v_1 v_2 \ldots$$

such that $Cl(u_i) \subseteq Cl(v_i)$ *for all* i *and* $|u_j|, |v_j|$ *are uniformly bounded by a constant* C*. Call these presentations of* α *and* β **witnessing partitions** *and intervals* u_i *and* v_i **partitioning intervals***.*

Theorem

$\alpha \leq_{QI} \beta$ *implies* $\alpha \leq_{CR} \beta$*.*

If the QI map is monotonic, then the proof is easy. It is not in the non-monotonic case.
We use a refined version of decomposition theorem and show a transitivity-like lemma.
A function of the following form is called an *atomic crossing map*:



Proof idea

Lemma

Any bijective quasi-isometry can be decomposed into finite number of atomic crossing maps, each of which are also quasi-isometry.

Lemma

Suppose $\alpha \leq_{QI} \beta$ *via an atomic crossing map* $f : \alpha \to \beta$ *and* $\beta \leq_{CR} \gamma$*. Then* $\alpha \leq_{CR} \gamma$*.*

($\alpha \leq_{QI} \beta$ implies $\alpha \leq_{CR} \beta$.) Decompose the QI map into $\alpha \xrightarrow{f_1} \gamma \xrightarrow{f_2} \beta$, where f_1 is bijective and f_2 is monotonic. Then apply the lemma above iteratively.

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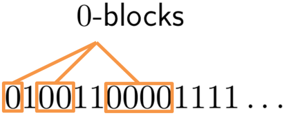
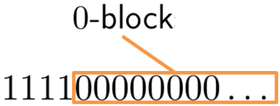
Notations

- From now on we assume $\Sigma = \{0, 1\}$.
- For $\alpha = 0^{n_0}1^{m_0}0^{n_1}1^{m_1}\ldots \in \{0,1\}^{\omega}(n_i, m_i \geq 1)$, we call 0^{n_i} and 1^{m_i} the 0-**blocks and** 1-**blocks**, respectively.



- An infinite succession of $\sigma \in \Sigma$ is also called a σ-block.



Global nature of Σ_{QI}^{ω}

We split the set Σ_{QI}^{ω} into four subsets:

- $\mathcal{X}(0) = \{[\alpha] \mid$ in α all the lengths of 0-blocks are universally bounded$\}$,
- $\mathcal{X}(1) = \{[\alpha] \mid$ in α the lengths of all 1-blocks are universally bounded$\}$,
- $\mathcal{X}(u) = \{[\alpha] \mid$ in α the lengths of both 0-blocks and 1-blocks are unbounded$\}$,
- $\mathcal{X}(b) = \{[\alpha] \mid$ in α the lengths of both 0-blocks and 1-blocks are universally bounded$\}$.

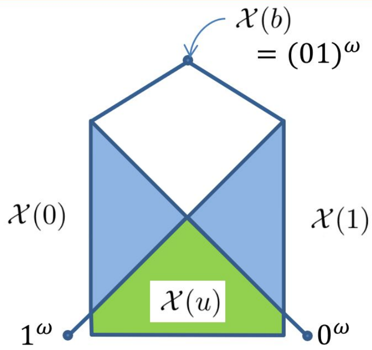
Theorem

The sets $\mathcal{X}(0)$, $\mathcal{X}(1)$, $\mathcal{X}(u)$, $\mathcal{X}(b)$ have the following properties:

1. *The sets $\mathcal{X}(0)$ and $\mathcal{X}(1)$ are filters.*
2. *The set $\mathcal{X}(u)$ is an ideal.*
3. *The set $\mathcal{X}(b)$ is the singleton $\{[(01)^{\omega}]\}$.*

Structure theorems

- The set $\mathcal{X}(b)$ is the singleton $\{[(01)^\omega]\}$, and is the greatest element.
- The sets $\mathcal{X}(0)$ and $\mathcal{X}(1)$ are filters.
- The set $\mathcal{X}(u)$ is an ideal.
- $[0^\omega]$ and $[1^\omega]$ are minimal.



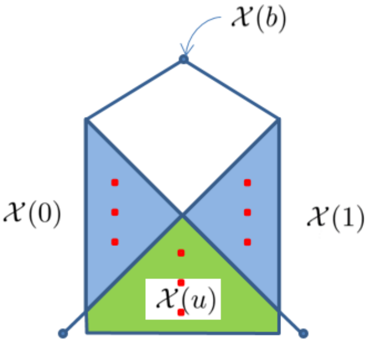
- $\mathcal{X}(0), \mathcal{X}(1)$ and $\mathcal{X}(u)$ contain chains $(\alpha_n)_{n\in\mathbb{Z}}$ of the type of integers, that is $\forall n \in \mathbb{Z}[\alpha_n <_{QI} \alpha_{n+1}]$.

Proof:

$$\alpha_1 = 0101001001\ldots0^{2^n}10^{2^n}1\ldots$$

$$\alpha_0 = 01001\ldots0^{2^n}1\ldots$$

$$\alpha_{-1} = 0100001\ldots0^{4^n}1\ldots$$

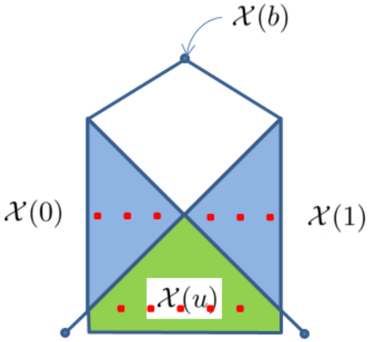


Structure theorems

- $\mathcal{X}(0), \mathcal{X}(1)$ and $\mathcal{X}(u)$ have countable antichains.

Proof:

$$\beta_n = 010^{2^n}1^{2^n}0^{3^n}1^{3^n}...0^{k^n}1^{k^n}...$$



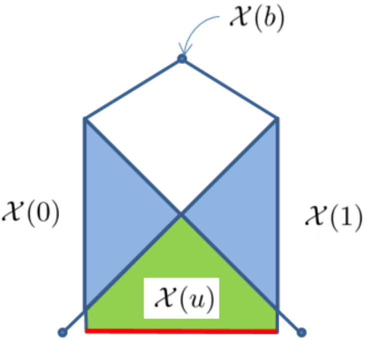
- Σ^{ω}_{QI} possesses infinitely many minimal elements.

Proof. For any unbounded nondecreasing sequence $\{a_n\}_{n\in\omega}$, the following sequence is minimal:

$$\alpha = 0^{a_0}1^{a_1}0^{a_2}1^{a_3}...0^{a_{2k}}1^{a_{2k+1}}\dots$$

Problem

Are there uncountably many minimal elements?



Density

Theorem (Density Theorem)

Let $\alpha, \beta \in \Sigma^\omega$ be given. Assume $\alpha <_{QI} \beta$, and every letter in α or β occurs in both α and β infinitely many often. Then there exists $\gamma \in \Sigma^\omega$ such that $\alpha <_{QI} \gamma <_{QI} \beta$.
Morever, there are infinitely many γ's that satisfy this inequality, and not quasi-isometric each other.

Least upper bound

Some naive definitions turn out to be not well-defined (i.e. there are $\alpha \sim_{QI} \alpha'$ and $\beta \sim_{QI} \beta'$ such that $[\alpha \wedge \beta] \neq [\alpha' \wedge \beta']$).

- $\alpha \wedge \beta = \alpha(0)\beta(0)\alpha(1)\beta(1)\dots$
- $\alpha \wedge \beta = \alpha \text{ XOR } \beta$

Theorem (Stephan+, personal communication)

There are strings α and β for which no least upper bound exist.

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Atlas

Definition

An **atlas** *is a set of quasi-isometry types. In particular, the atlas defined by the language L is the set $[L] = \{[\alpha] \mid \alpha \in L\}$, where $[\alpha]$ is the quasi-isometry type of α.*

Definition

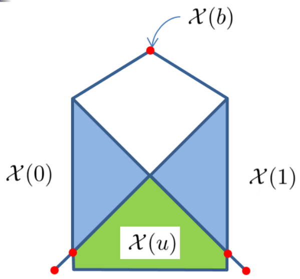
A **Büchi automaton** $\mathcal{M}$ *is a quadruple (S, ι, Δ, F), where S is a finite set of states, $\iota \in S$ is the initial state, $\Delta \subset S \times \Sigma \times S$ is the transition table, and $F \subseteq S$ is the set of accepting states.*

Geometries of strings accepted by $\mathcal{M}$

Theorem

Any atlas $[L]$ defined by a Büchi recognisable language L is a union from the following list of atlases:

- $[\{(01)^\omega\}]$, $[\{1^\omega\}]$, $[\{0^\omega\}]$, $[\{01^\omega\}]$, $[\{10^\omega\}]$,
- $\Sigma^\omega_{QI} \setminus \{[0^\omega], [1^\omega], [10^\omega], [01^\omega]\}$,
- $\mathcal{X}(0) \setminus \{[1^\omega], [01^\omega]\}$,
- $\mathcal{X}(1) \setminus \{[0^\omega], [10^\omega]\}$.



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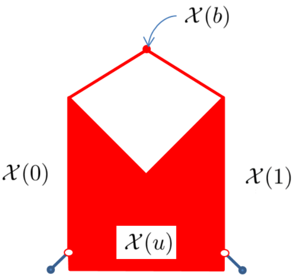

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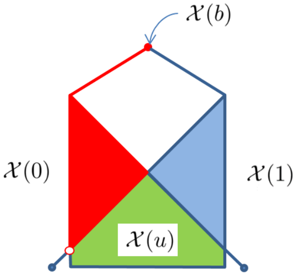

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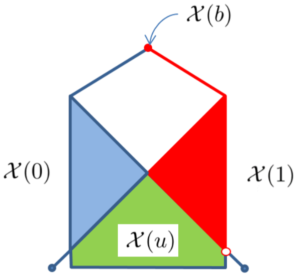

Proof idea

Call a loop of a Büchi automaton a *0-loop* if only 0 is read through the loop. Define *1-loops* and *01-loops* in a similar way.
Then all Büchi automata are categorized by the following features:

- if it has a 0-loop, 1-loop, and 01-loop or not; and
- how these loops are connected.

For example,

- if it has 0-loop and 1-loop, the initial state is in 0-loop and can move from one loop to another, then the automaton accepts $\Sigma_{QI}^{\omega} \setminus [1^{\omega}]$.
- If it has 0-loop and 01-loop, the initial state is in 0-loop and can move from one loop to another, then the automaton accepts $\mathcal{X}(1)$.

Decidability result

Corollary

There exists an algorithm that, given Büchi automata $\mathcal{A}$ and $\mathcal{B}$, decides if the atlases $[L(\mathcal{A})]$ and $[L(\mathcal{B})]$ coincide. Furthermore, the algorithm runs in linear time on the size of the input automata.

In contrast, the problem of deciding whether two given Büchi automata represent the same language is PSPACE-complete.

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Problem and our result

The quasi-isometry problem consists of determining if given two strings α and β are quasi-isometric. Formally, the quasi-isometry problem (over the alphabet Σ) is identified as the set:

$$QIP = \{(\alpha, \beta) \mid \alpha, \beta \in \Sigma^\omega \ \& \ [\alpha] = [\beta]\}.$$

Theorem

The following statements are true:

1. *Given quasi-isometric strings α and β, there exists a quasi-isometry between α and β computable in the halting set relative to α and β.*
2. *The quasi-isometry problem between computable strings, that is the following set $QIP = \{(\alpha, \beta) \mid \alpha, \beta \in \Sigma^\omega, [\alpha] = [\beta]$, α and β are computable$\}$ is a complete Σ^0_2-set.*

Problem

Given quasi-isometric strings α and β, does there exist a computable quasi-isometry between them?

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Basic setting

- Let $\alpha \in \Sigma^\omega$ be a coloured metric space.
- We call $s : \omega \to \omega$ a *scaling factor* if it is strictly monotonic and $s(0) = 1$.
- Let $d_n(i,j) = |i - j|/s(n)$.

We define the following sequence of metric spaces:

$$X_{0,\alpha} = (\alpha, d_0),\ X_{1,\alpha} = (\alpha, d_1), \ldots, X_{n,\alpha} = (\alpha, d_n), \ldots$$

We want to define a "limit" of this sequence in a formal way to treat the large scale geometry of α. We adopt the notion of asymptotic cone to do that.

The set $\mathbf{B}(\mathcal{F}, s)$ and an equiv. rel. on it

- Let $\mathcal{F} \subset P(\omega)$ be a non-principal ultrafilter.
- Let $\mathbf{a} = (a_n)_{n \geq 0}$ be a sequence, where $a_n \in X_{n,\alpha}$.
- $\mathbf{a}$ is *$\mathcal{F}$-bounded* if $\{n \mid d_n(0, a_n) < L\} \in \mathcal{F}$ for some L.
- Let $\mathbf{B}(\mathcal{F}, s)$ be the set of all bounded sequences $\mathbf{a} = (a_n)_{n \geq 0}$.
- **a**, **b** $\in \mathbf{B}(\mathcal{F}, s)$ is said to be *$\mathcal{F}$-equivalent* ($\mathbf{a} \sim_{\mathcal{F}} \mathbf{c}$) if

$$\forall \epsilon > 0[\{n \mid d_n(a_n, b_n) \leq \epsilon\} \in \mathcal{F}].$$

Asymptotic cone

Definition

For given sequence α, scaling function s and ultrafilter $\mathcal{F}$, the **asymptotic cone of** α*, written $Cone(\alpha, \mathcal{F}, s)$, with respect to the scaling function $s(n)$ and the ultra-filter $\mathcal{F}$ is the factor set*

$$\mathbf{B}(\mathcal{F}, s)/\sim_{\mathcal{F}}$$

equipped with the following metric D and colour C:

1. $D(\mathbf{a}, \mathbf{b}) = r$ *if and only if for every ϵ the set $\{n \mid r - \epsilon \leq d_n(a_n, b_n) \leq r + \epsilon\}$ belong to $\mathcal{F}$.*
2. $C(\mathbf{a}) = \sigma$ *if and only if the set $\{n \mid a_n$ has colour $\sigma\}$ belongs to $\mathcal{F}$.*

Results

The following theorems are coloured variant of a known results in geometric group theory, which says the set of all asymptotic cones modulo quasi-isometry is "simpler" than Σ^{ω}_{QI}.

Theorem

If strings α and β are quasi-isometric then the following holds for the asymptotic cones $Cone(\alpha, \mathcal{F}, s)$ and $Cone(\beta, \mathcal{F}, s)$.

1. *They are bi-Lipschitz equivalent; i.e. they are quasi-isometric with the additive constant $B = 0$.*
2. *The bi-Lipscitz map above can be taken as a order preserving map.*

Theorem

There are two non-quasi-isometric strings $\alpha, \beta \in \{0, 1\}^{\omega}$, a scale factor $s(n)$, and filter $\mathcal{F}$ such that the cones $Cone(\alpha, \mathcal{F}, s)$ and $Cone(\beta, \mathcal{F}, s)$ coincide.

Theorem

If α is Martin-Löf random, then for all computable scaling factors s and ultra-filters $\mathcal{F}$, the asymptotic cone $Cone(\alpha, \mathcal{F}, s)$ coincides with the space $(\mathcal{R}_{\geq 0}; d, C)$, where all reals have all colours from alphabet Σ.

Future work

- Open problems
  - Cardinality of the set of minimal elements
  - Existence of computable QI-map for computable sequences
  - There are some more...
- Degree theory for $\leq_{QI}$ (ongoing w/ F. Stephan, S. Jain)