Rogers Semilattices and Their First-Order Theories

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Numberings

Let S be a countable set. A **numbering** ν of the set S is a surjective map from the set of natural numbers $\mathbb{N}$ onto S.

Suppose that ν is a numbering of a set S_0, and μ is a numbering of a set S_1. The numbering ν is **reducible** to μ, denoted by $\nu \leq \mu$, if there is a total computable function $f(x)$ such that

$$\nu(n) = \mu(f(n)) \text{ for all } n \in \mathbb{N}.$$

Note the following: if $\nu \leq \mu$, then $S_0 \subseteq S_1$.

Numberings ν and μ are **equivalent** (denoted by $\nu \equiv \mu$) if $\nu \leq \mu$ and $\mu \leq \nu$.

Computable families

In the talk, we consider only numberings of families $S \subset P(\mathbb{N})$.

A numbering ν is **computable** if the set

$$\{\langle n, x\rangle : x \in \nu(n)\}$$

is computably enumerable.

A family $S \subset P(\mathbb{N})$ is *computable* if S has a computable numbering.

By $Com^0_1(S)$ we denote the set of all computable numberings of the family S.

Rogers semilattices for computable families

Let S be a computable family. Given numberings ν and μ of S, one defines a new numbering $\nu \oplus \mu$ as follows.

$$(\nu \oplus \mu)(2n) := \nu(n), \quad (\nu \oplus \mu)(2n+1) := \mu(n).$$

The quotient structure

$$\mathcal{R}^0_1(S) := (Com^0_1(S); \leq, \oplus)/_{\equiv}$$

is an upper semilattice. It is called the **Rogers semilattice** of the computable family S.

Example (from [Ershov 1977]).

Let $A \neq B$ be c.e. sets. Consider the family $S = \{A, B\}$. What can one say about the Rogers semilattice $\mathcal{R}^0_1(S)$?

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Let $A \neq B$ be c.e. sets. Consider the family $S = \{A, B\}$. What can one say about the Rogers semilattice $\mathcal{R}^0_1(S)$?

Case 1. Assume that $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$. Choose elements $a \in A \setminus B$ and $b \in B \setminus A$.

Let ν be a computable numbering of S. Then for any $n \in \mathbb{N}$, we have:

$$\nu(n) = A \;\Leftrightarrow\; a \in \nu(n) \;\Leftrightarrow\; b \notin \nu(n).$$

Hence, the set $\nu^{-1}[A] := \{n \in \mathbb{N} : \nu(n) = A\}$ is computable.

This fact implies that *any* computable numberings ν and μ of the family S are equivalent.

In other words, the semilattice $\mathcal{R}^0_1(S)$ is *one-element*.

Case 2. Suppose that $A \subset B$ or $B \subset A$. W.l.o.g., one may assume that $A \subset B$ and $b \in B \setminus A$.

Let W be a c.e. set such that $W \neq \emptyset$ and $W \neq \mathbb{N}$. One can define a numbering

$$\nu^W(n) := \begin{cases} A, & \text{if } n \notin W, \\ B, & \text{if } n \in W. \end{cases}$$

It is not hard to see that ν^W is a computable numbering of S. Furthermore, for any c.e. sets W and V,

$$\nu^W \leq \nu^V \ \Leftrightarrow \ W \leq_m V.$$

On the other hand, if μ is an arbitrary computable numbering of S, then $(\mu(n) = B) \Leftrightarrow (b \in \mu(n))$. Thus, the set $\mu^{-1}[B]$ is c.e.

Therefore, we deduce that the structure $\mathcal{R}_1^0(S)$ is *isomorphic* to the semilattice of c.e. m-degrees $\mathbf{R}_m$.

The study of computable numberings goes back to 1950s (Friedberg, Kleene, Kolmogorov, Rogers, Uspenskii, . . .). The monograph of Ershov (1977) became one of the keystones of the theory of numberings.

Here we talk about two directions:

(i) Numberings in the hyperarithmetical and analytical hierarchies.

(ii) Different kinds of reducibilities between numberings.

Γ-Computable numberings

Goncharov and Sorbi (1997) developed the foundations of the theory of generalized computable numberings.

Let Γ be a complexity class (e.g., Σ^0_1, Π^0_2, or d-Σ^0_1). A numbering ν of a family $S \subset P(\mathbb{N})$ is Γ-**computable** if

$$\{\langle n, x\rangle : x \in \nu(n)\} \in \Gamma.$$

A family S is Γ-*computable* if S has a Γ-computable numbering. By $Com_\Gamma(S)$ we denote the set of all Γ-computable numberings of S.

Note that the classical notion of a computable numbering becomes a synonym for a Σ^0_1-computable numbering.

Γ-Computable numberings

Assume that a class Γ has the following properties:

(a) If ν is a Γ-computable numbering and μ is a numbering such that $\mu \leq \nu$, then μ is also Γ-computable.

(b) If numberings ν and μ are both Γ-computable, then the numbering $\nu \oplus \mu$ is also Γ-computable.

[It is not hard to show that any Σ- or Π-class from standard recursion-theoretic hierarchies (e.g., arithmetical, hyperarithmetical, analytical, or the Ershov hierarchy) satisfies these conditions.]

The quotient structure

$$\mathcal{R}_\Gamma(S) := (Com_\Gamma(S); \leq, \oplus)/_\equiv$$

is an upper semilattice. It is called the **Rogers semilattice** of the Γ-computable family S.

We say that an upper semilattice $\mathcal{U} = (U; \leq, \vee)$ is a **Rogers Γ-semilattice** if there is a Γ-computable family $S \subset P(\mathbb{N})$ such that $\mathcal{U} \cong \mathcal{R}_\Gamma(S)$.

Problem 1

Study isomorphism types of Rogers Γ-semilattices.

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Outline:

(a) Some global properties of Rogers semilattices.

(b) Comparing isomorphism types of Rogers semilattices for different Γ.

(c) Number of isomorphism types (for a fixed Γ).

(d) Elementary theories.

Some global properties of Rogers semilattices

Theorem (Khutoretskii 1971)

Every Rogers Σ^0_1-semilattice is either one-element, or infinite.

Theorem (Selivanov 1976)

If a Rogers Σ^0_1-semilattice is infinite, then it is not a lattice.

Some global properties of Rogers semilattices

Theorem (Goncharov and Sorbi 1997)

Let $\alpha \geq 2$ be a computable successor ordinal. Let S be a Σ^0_α-computable family such that S contains at least two elements. Then the Rogers semilattice $\mathcal{R}_{\Sigma^0_\alpha}(S)$ is infinite, and it is not a lattice.

Note that if S contains precisely one element, then the structure $\mathcal{R}_{\Sigma^0_\alpha}(S)$ is one-element.

Theorem (essentially, Dorzhieva 2016)

Let n be a non-zero natural number. Let S be a Π^1_n-computable family with at least two elements. Then the semilattice $\mathcal{R}_{\Pi^1_n}(S)$ is infinite, and it is not a lattice.

Rogers semilattices for different levels

We say that an upper semilattice is *non-trivial* if it contains more than one element.

Theorem (Podzorov 2008)

Suppose that m and n are non-zero natural numbers with $m \geq n + 2$. If $\mathcal{U}$ is a non-trivial Rogers Σ^0_m-semilattice, then $\mathcal{U}$ is not isomorphic to any Rogers Σ^0_n-semilattice.

Theorem (Badaev and Goncharov 2008)

Suppose that α and β are non-zero computable ordinals such that $\alpha \geq \beta + 3$. If $\mathcal{U}$ is a non-trivial Rogers Σ^0_α-semilattice, then $\mathcal{U}$ is not isomorphic to any Rogers Σ^0_β-semilattice.

Rogers semilattices for different levels

Theorem 1 (B., Ospichev, and Yamaleev)

Suppose that m and n are non-zero natural numbers with $m \geq n + 1$. If $\mathcal{U}$ is a non-trivial Rogers Π^1_m-semilattice, then $\mathcal{U}$ is not isomorphic to any Rogers Π^1_n-semilattice.

Proof of Theorem 1: Outline

(1) First, we give an upper bound on the complexity of intervals inside Rogers Π^1_n-semilattices.

By E_n we denote the m-complete Σ^1_n set.

Lemma 1

Let $\mathcal{M}$ be a Rogers Π^1_n-semilattice. If $a <_{\mathcal{M}} b$ are elements from $\mathcal{M}$, then the interval $[a;b]_{\mathcal{M}}$, treated as a structure in the language $\{\leq, \vee\}$, has a $\Sigma^0_2(E_n)$ presentation.

Proof of Theorem 1: Outline

Recall that $m \geq n+1$.

(2) We show that inside any non-trivial Rogers Π^1_m-semilattice, one can find a "sufficiently complicated" interval.

By applying the result of Downey and Knight (1992), one can choose a countable linear order $\mathcal{L}$ such that:

- $\mathcal{L}$ has the least and the greatest elements,
- $\mathcal{L}$ has a $\deg_T(E_n)^{(5)}$-computable copy, and
- $\mathcal{L}$ has no $\Sigma^0_2(E_n)$-computable presentations.

By Lemma 1, $\mathcal{L}$ cannot be realized as an interval inside a Rogers Π^1_n-semilattice.

On the other hand, $\mathcal{L}$ has a Δ^1_{n+1} presentation, and this is enough to obtain the following: Inside any non-trivial Rogers Π^1_m-semilattice, there exists an interval isomorphic to $\mathcal{L}$.

Levels of the Ershov hierarchy

Let n be a non-zero natural number. Recall that:

- A set $X \subseteq \mathbb{N}$ is $2n$-**c.e.** if there are c.e. sets $W_1, W_2, \dots, W_{2n-1}, W_{2n}$ such that

$$X = (W_1 \setminus W_2) \cup (W_3 \setminus W_4) \cup \dots \cup (W_{2n-1} \setminus W_{2n}).$$

- A set X is $(2n+1)$-**c.e.** if there are c.e. sets $W_1, W_2, \dots, W_{2n-1}, W_{2n}, W_{2n+1}$ such that

$$X = (W_1 \setminus W_2) \cup (W_3 \setminus W_4) \cup \dots \cup (W_{2n-1} \setminus W_{2n}) \cup W_{2n+1}.$$

By Σ^{-1}_m we denote the class of all m-c.e. sets.

Levels of the Ershov hierarchy

The case of the Ershov hierarchy is quite different:

Theorem (Herbert, Jain, Lempp, Mustafa, and Stephan)

Let m and n be non-zero natural numbers with $m \geq n+1$. Then every Rogers Σ^{-1}_n-semilattice is also a Rogers Σ^{-1}_m-semilattice

Number of isomorphism types: Σ^0_1 case

Theorem (Ershov and Lavrov 1973)

There are finite families F_i, $i \in \omega$, of c.e. sets such that the Rogers semilattices $\mathcal{R}_{\Sigma^0_1}(F_i)$, $i \in \omega$, are pairwise non-isomorphic.

Theorem (V'yugin 1973)

There are Σ^0_1-computable families S_i, $i \in \omega$, such that the semilattices $\mathcal{R}_{\Sigma^0_1}(S_i)$, $i \in \omega$, are pairwise elementarily non-equivalent.

Number of isomorphism types

Theorem (Badaev, Goncharov, and Sorbi 2005)

Let $n \geq 2$ be a natural number. There are Σ^0_n-computable families S_i, $i \in \omega$, such that the semilattices $\mathcal{R}_{\Sigma^0_n}(S_i)$, $i \in \omega$, are pairwise elementarily non-equivalent.

Open Question 1

Let n be a non-zero natural number. Are there infinitely many pairwise non-isomorphic Rogers Π^1_n-semilattices?

Elementary theories of Rogers Γ-semilattices

Theorem (Badaev, Goncharov, Podzorov, and Sorbi 2003)

Let $n \geq 2$ be a natural number. For any non-trivial Rogers Σ^0_n-semilattice $\mathcal{U}$, its first-order theory $Th(\mathcal{U})$ is hereditarily undecidable.

Theorem (Dorzhieva 2016)

Let n be a non-zero natural number. For any non-trivial Rogers Π^1_n-semilattice $\mathcal{U}$, the theory $Th(\mathcal{U})$ is hereditarily undecidable.

Let $\mathcal{U}$ be a non-trivial Rogers Π^1_n-semilattice.

Dorzhieva (2016) provides a first-order interpretation (with parameters) of the poset $(\mathcal{E}^*; \subseteq^*)$ inside $\mathcal{U}$. An analysis of the interpretation shows the following:

Corollary

The set Π_6-$Th_{\leq}(\mathcal{U})$, i.e. the Π_6-fragment of the theory $Th(\mathcal{U})$ in the language $\{\leq\}$, is hereditarily undecidable.

The methods of the proof of Theorem 1 allow to obtain the following:

Proposition

- The fragment Σ_1-$Th_{\leq}(\mathcal{U})$ is decidable.
- The fragment Π_3-$Th_{\leq}(\mathcal{U})$ is hereditarily undecidable.

Problem 2

Study the complexity of the first-order theories for Rogers Γ-semilattices: What are their m-degrees?

In general, the problem seems to be quite hard to attack: e.g., recall that there are countably many elementary theories of Rogers Σ^0_1-semilattices [V'yugin 1973].

As a first step to tackle Problem 2, we introduce another approach.

The semilattice $\mathcal{R}$ of all numberings

Recall that in general, when we say that a numbering ν is reducible to a numbering μ, we *do not require* that ν and μ are numberings of the same family. Indeed,

$$\left.\begin{array}{l} \nu\colon \mathbb{N} \underset{\text{onto}}{\longrightarrow} S, \\ \mu\colon \mathbb{N} \underset{\text{onto}}{\longrightarrow} T, \\ \nu \leq \mu \end{array}\right\} \Rightarrow \ S \subseteq T.$$

Therefore, one can consider the following structure:

Let $Num := \{\nu : \nu$ is a function from $\mathbb{N}$ into $P(\mathbb{N})\}$. Then the quotient

$$\mathcal{R} := (Num; \leq, \oplus)/_{\equiv}$$

is well-defined, and it is an upper semilattice. We say that $\mathcal{R}$ is the *Rogers semilattice of all numberings*.

One can also work with natural subsystems of $\mathcal{R}$:

Let Γ be a complexity class (e.g., Σ^0_2 or Π^1_1). Then the structure

$$\mathcal{R}_\Gamma := (\{\nu \in Num : \nu \text{ is } \Gamma\text{-computable}\}; \leq, \oplus)/_\equiv$$

is called the *Rogers semilattice of all* Γ*-computable numberings*.

The complexity of $Th(\mathcal{R}_{\Sigma^0_1})$

Theorem 2 (B., Mustafa, and Yamaleev 2019)

The theory $Th(\mathcal{R}_{\Sigma^0_1})$, i.e. the elementary theory of the Rogers semilattice of all Σ^0_1-computable numberings, is recursively isomorphic to first-order arithmetic.
Moreover, the fragment Π_5-$Th(\mathcal{R}_{\Sigma^0_1})$ is hereditarily undecidable.

Proof of Theorem 2: Outline

(1) Recall the example from [Ershov 1977]: If $A \subsetneq B$ are c.e. sets, then the Rogers Σ^0_1-semilattice $\mathcal{R}_{\Sigma^0_1}(\{A, B\})$ is isomorphic to the semilattice of c.e. m-degrees $\mathbf{R}_m$.

This immediately gives a first-order interpretation (with parameters!) of $\mathbf{R}_m$ inside $\mathcal{R}_{\Sigma^0_1}$.

(2) In order to eliminate the parameters from the interpretation, we employ the following simple fact: Every minimal element inside $\mathcal{R}_{\Sigma^0_1}$ is induced by a numbering of a one-element family. Thus, one can also provide a first-order definition for the elements induced by numberings of two-element families.

Recall than Nies (1994) proved that the theory $Th(\mathbf{R}_m)$ is recursively isomorphic to first-order arithmetic. Since $\mathbf{R}_m$ is first-order interpretable without parameters inside $\mathcal{R}_{\Sigma^0_1}$, we deduce that $Th(\mathcal{R}_{\Sigma^0_1}) \equiv_1 Th(\mathbf{R}_m)$.

The complexity of $Th(\mathcal{R})$

Recall that $\mathcal{R}$ is the semilattice of all numberings.

Theorem 3 (B., Mustafa, and Yamaleev 2019)

The theory $Th(\mathcal{R})$ is recursively isomorphic to second-order arithmetic. The fragment Π_5-$Th(\mathcal{R})$ is hereditarily undecidable.

The proof is similar to the previous one, modulo the following modification: We use that the elementary theory of the semilattice of all m-degrees $\mathbf{D}_m$ is recursively isomorphic to second-order arithmetic [Nerode and Shore 1980].

The complexity of $Th(\mathcal{R}_{\Sigma^0_\alpha})$

We also obtain a partial result on the semilattice of all Σ^0_α-computable numberings.

Proposition (B., Mustafa, and Yamaleev 2019)

Let α be a computable successor ordinal such that $\alpha \geq 2$. The elementary theory of the semilattice of Σ^0_α m-degrees is 1-reducible to $Th(\mathcal{R}_{\Sigma^0_\alpha})$.

Open Question 2

Let α be a non-zero computable ordinal. Is it true that for any non-trivial Rogers Σ^0_α-semilattices $\mathcal{U}$ and $\mathcal{V}$, the theories $Th(\mathcal{U})$ and $Th(\mathcal{V})$ are recursively isomorphic?

Note that the following question is still open:

Is it true that for any non-trivial Rogers Σ^0_1-semilattice $\mathcal{U}$, the theory $Th(\mathcal{U})$ is undecidable?

In the setting of analytical numberings, the following problem seems to be of interest:

Open Question 3

Study the connections between additional set-theoretic assumptions and the results on Π^1_n-computable numberings.

So far, there are only few results in this direction.

Friedberg numberings

A numbering ν is *Friedberg* if for any natural numbers $m \neq n$, we have $\nu(m) \neq \nu(n)$.

Theorem (Owings 1970)

The family of all Π^1_1 sets has no Π^1_1-computable Friedberg numbering.

Theorem (Dorzhieva)

(a) The family of all Π^1_2 sets has no Π^1_2-computable Friedberg numbering.

(b) Assume $V = L$. For any natural number $n \geq 3$, the family of all Π^1_n sets has no Π^1_n-computable Friedberg numbering.

II. Different kinds of reducibility between numberings:

(a) Relativized reducibility.

(b) bm-Reducibility.

Relativized reducibility

Let $\mathbf{d}$ be a Turing degree.

A numbering ν is $\mathbf{d}$-*reducible* to a numbering μ, denoted by $\nu \leq_{\mathbf{d}} \mu$, if there is a $\mathbf{d}$-computable function $f(x)$ such that $\nu(n) = \mu(f(n))$ for all n.

For a Γ-computable family S, one can introduce an upper semilattice

$$\mathcal{R}_{\Gamma}^{\mathbf{d}}(S) := (Com_{\Gamma}(S); \leq_{\mathbf{d}}, \oplus)/_{\equiv_{\mathbf{d}}}.$$

$\mathbf{0}^{(n)}$-reducibility

In the series of papers (2003), Badaev, Goncharov, Podzorov, and Sorbi obtained several results on semilattices $\mathcal{R}^{\mathbf{0}^{(n)}}_{\Sigma^0_m}(S)$, where $n, m \in \omega$: in particular,

Theorem (Badaev, Goncharov, and Sorbi 2003)

Let m be a non-zero natural number. Consider the following two Σ^0_m-computable families:

- F is the family of all finite sets,
- S is the family of all Σ^0_m sets.

Then for any $i < m$, the semilattices $\mathcal{R}^{\mathbf{0}^{(i)}}_{\Sigma^0_m}(F)$ and $\mathcal{R}^{\mathbf{0}^{(i)}}_{\Sigma^0_m}(S)$ are not isomorphic.

Theorem (Goncharov 1982)

Let S be a Σ^0_1-computable family. Suppose that S has two Σ^0_1-computable Friedberg numberings ν and μ such that $\nu \not\equiv \mu$ and $\nu \equiv_{\mathbf{0}'} \mu$. Then S has infinitely many pairwise non-equivalent, Σ^0_1-computable Friedberg numberings.

This result has a counterpart in computable structure theory:

Theorem (Goncharov 1982)

Let $\mathcal{A}$ be a computable structure. Suppose that $\mathcal{B}$ is a computable copy of $\mathcal{A}$ such that $\mathcal{B} \not\cong_{\Delta^0_1} \mathcal{A}$ and $\mathcal{B} \cong_{\Delta^0_2} \mathcal{A}$. Then $\mathcal{A}$ has infinite computable dimension.

bm-Reducibility

Maslova (1979) introduced a *bounded version* of m-reducibility on sets:

Let A and B be subsets of $\mathbb{N}$. Then $A \leq_{bm} B$ if there is a total computable function $g(x)$ such that:

(a) $n \in A$ iff $g(n) \in B$, for all $n \in \mathbb{N}$;

(b) for any $k \in \mathbb{N}$, the preimage $g^{-1}(k)$ is a finite set.

Following her approach, we say that a numbering ν is bm-**reducible** to a numbering μ, denoted by $\nu \leq_{bm} \mu$, if there is a total computable function $f(x)$ with the following properties:

1. $\nu(n) = \mu(f(n))$, for all $n \in \mathbb{N}$;
2. for any $k \in \mathbb{N}$, the set $f^{-1}(k)$ is finite.

Consider a Γ-computable family S. In a natural way, the reducibility $\leq_{bm}$ gives rise to the corresponding *Rogers bm-semilattice*:

$$\mathcal{R}_{\Gamma}^{bm}(S) := (Com_{\Gamma}(S); \leq_{bm}, \oplus)/_{\equiv_{bm}}.$$

Cardinalities of bm-semilattices

Recall the result of Khutoretskii (1971):
For a Σ^0_1-computable family S, the semilattice $\mathcal{R}_{\Sigma^0_1}(S)$ is either one-element, or infinite.

This result *does not extend* to the case of $\leq_{bm}$:

Theorem 4 (B., Mustafa, and Ospichev)

- Let S be a finite Σ^0_1-computable family with precisely $n \geq 2$ elements. Then the cardinality of the Rogers bm-semilattice $\mathcal{R}^{bm}_{\Sigma^0_1}(S)$ is either equal to $2^n - 1$, or countably infinite. Both these cases can be realized.
- For any non-zero natural number n, there is an infinite Σ^0_1-computable family T such that the cardinality of the bm-semilattice $\mathcal{R}^{bm}_{\Sigma^0_1}(T)$ is equal to 2^n.

There is a bm-lattice

Recall the result of Selivanov (1976):
Let S be a Σ^0_1-computable family. If the structure $\mathcal{R}_{\Sigma^0_1}(S)$ is infinite, then it is not a lattice.

Again, this *cannot be extended* to the bm-case:

Proposition (B., Mustafa, and Ospichev)

Consider a Σ^0_1-computable family $S := \{\{k\} : k \in \mathbb{N}\}$. Then the structure $\mathcal{R}^{bm}_{\Sigma^0_1}(S)$ is isomorphic to the lattice of all Π^0_2 sets (under inclusion).

Nevertheless, our results above do not transfer to further levels of the hyperarithmetical hierarchy:

Proposition (B., Mustafa, and Ospichev)

Let $\alpha \geq 2$ be a computable successor ordinal. Let S be a Σ^0_α-computable family such that S contains at least two elements. Then the bm-semilattice $\mathcal{R}^{bm}_{\Sigma^0_\alpha}(S)$ is infinite, and it is not a lattice.

Open Question 4

Provide a complete characterization for possible cardinalities of bm-semilattices $\mathcal{R}^{bm}_{\Sigma^0_1}(S)$.

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