Rogers Semilattices and Their First-Order Theories

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Numberings

Let $S$ be a countable set. A **numbering** $\nu$ of the set $S$ is a surjective map from the set of natural numbers $\mathbb{N}$ onto $S$.

Suppose that $\nu$ is a numbering of a set $S_0$, and $\mu$ is a numbering of a set $S_1$. The numbering $\nu$ is **reducible** to $\mu$, denoted by $\nu \leq \mu$, if there is a total computable function $f(x)$ such that

$$\nu(n) = \mu(f(n)) \text{ for all } n \in \mathbb{N}.$$  

Note the following: if $\nu \leq \mu$, then $S_0 \subseteq S_1$.

Numberings $\nu$ and $\mu$ are **equivalent** (denoted by $\nu \equiv \mu$) if $\nu \leq \mu$ and $\mu \leq \nu$. 
In the talk, we consider only numberings of families $S \subset P(\mathbb{N})$.

A numbering $\nu$ is **computable** if the set

$$\{ \langle n, x \rangle : x \in \nu(n) \}$$

is computably enumerable.

A family $S \subset P(\mathbb{N})$ is **computable** if $S$ has a computable numbering.

By $Com^0_1(S)$ we denote the set of all computable numberings of the family $S$. 
Let $S$ be a computable family. Given numberings $\nu$ and $\mu$ of $S$, one defines a new numbering $\nu \oplus \mu$ as follows.

$$(\nu \oplus \mu)(2n) := \nu(n), \quad (\nu \oplus \mu)(2n + 1) := \mu(n).$$

The quotient structure

$$\mathcal{R}_1^0(S) := (Com_1^0(S); \leq, \oplus)/\equiv$$

is an upper semilattice. It is called the Rogers semilattice of the computable family $S$. 

Rogers semilattices for computable families
Example (from [Ershov 1977]).
Let \( A \neq B \) be c.e. sets. Consider the family \( S = \{ A, B \} \).
What can one say about the Rogers semilattice \( \mathcal{R}_1^0(S) \)?
Example (from [Ershov 1977]).

Let $A \neq B$ be c.e. sets. Consider the family $S = \{A, B\}$. What can one say about the Rogers semilattice $R_1^0(S)$?

**Case 1.** Assume that $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$. Choose elements $a \in A \setminus B$ and $b \in B \setminus A$.

Let $\nu$ be a computable numbering of $S$. Then for any $n \in \mathbb{N}$, we have:

$$\nu(n) = A \iff a \in \nu(n) \iff b \notin \nu(n).$$

Hence, the set $\nu^{-1}[A] := \{n \in \mathbb{N} : \nu(n) = A\}$ is computable.

This fact implies that *any* computable numberings $\nu$ and $\mu$ of the family $S$ are equivalent.

In other words, the semilattice $R_1^0(S)$ is *one-element*. 
Case 2. Suppose that $A \subset B$ or $B \subset A$. W.l.o.g., one may assume that $A \subset B$ and $b \in B \setminus A$.

Let $W$ be a c.e. set such that $W \neq \emptyset$ and $W \neq \mathbb{N}$. One can define a numbering

$$\nu^W(n) := \begin{cases} A, & \text{if } n \notin W, \\ B, & \text{if } n \in W. \end{cases}$$

It is not hard to see that $\nu^W$ is a computable numbering of $S$. Furthermore, for any c.e. sets $W$ and $V$,

$$\nu^W \leq \nu^V \iff W \leq_m V.$$  

On the other hand, if $\mu$ is an arbitrary computable numbering of $S$, then $(\mu(n) = B) \iff (b \in \mu(n))$. Thus, the set $\mu^{-1}[B]$ is c.e.

Therefore, we deduce that the structure $\mathcal{R}_1^0(S)$ is isomorphic to the semilattice of c.e. $m$-degrees $\mathbb{R}_m$. 

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The study of computable numberings goes back to 1950s (Friedberg, Kleene, Kolmogorov, Rogers, Uspenskii, . . .). The monograph of Ershov (1977) became one of the keystones of the theory of numberings.

Here we talk about two directions:

(i) Numberings in the hyperarithmetic and analytical hierarchies.

(ii) Different kinds of reducibilities between numberings.
Γ-Computable numberings

Goncharov and Sorbi (1997) developed the foundations of the theory of generalized computable numberings.

Let $\Gamma$ be a complexity class (e.g., $\Sigma^0_1$, $\Pi^0_2$, or $d-\Sigma^0_1$). A numbering $\nu$ of a family $S \subset P(\mathbb{N})$ is $\Gamma$-computable if

$$\{\langle n, x \rangle : x \in \nu(n) \} \in \Gamma.$$  

A family $S$ is $\Gamma$-computable if $S$ has a $\Gamma$-computable numbering. By $Com_\Gamma(S)$ we denote the set of all $\Gamma$-computable numberings of $S$.

Note that the classical notion of a computable numbering becomes a synonym for a $\Sigma^0_1$-computable numbering.
Γ-Computable numberings

Assume that a class Γ has the following properties:

(a) If ν is a Γ-computable numbering and μ is a numbering such that μ ≤ ν, then μ is also Γ-computable.

(b) If numberings ν and μ are both Γ-computable, then the numbering ν ⊕ μ is also Γ-computable.

[It is not hard to show that any Σ- or Π-class from standard recursion-theoretic hierarchies (e.g., arithmetical, hyperarithmetical, analytical, or the Ershov hierarchy) satisfies these conditions.]

The quotient structure

\[\mathcal{R}_\Gamma(S) := (\text{Com}_\Gamma(S); \leq, \oplus)/\equiv\]

is an upper semilattice. It is called the Rogers semilattice of the Γ-computable family S.
We say that an upper semilattice $\mathcal{U} = (U; \leq, \lor)$ is a Rogers $\Gamma$-semilattice if there is a $\Gamma$-computable family $S \subset P(\mathbb{N})$ such that $\mathcal{U} \cong \mathcal{R}_\Gamma(S)$.

Problem 1
Study isomorphism types of Rogers $\Gamma$-semilattices.
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Outline:

(a) Some global properties of Rogers semilattices.
(b) Comparing isomorphism types of Rogers semilattices for different $\Gamma$.
(c) Number of isomorphism types (for a fixed $\Gamma$).
(d) Elementary theories.
Some global properties of Rogers semilattices

Theorem (Khutoretskii 1971)
Every Rogers $\Sigma^0_1$-semilattice is either one-element, or infinite.

Theorem (Selivanov 1976)
If a Rogers $\Sigma^0_1$-semilattice is infinite, then it is not a lattice.
Some global properties of Rogers semilattices

Theorem (Goncharov and Sorbi 1997)

Let $\alpha \geq 2$ be a computable successor ordinal. Let $S$ be a $\Sigma^0_\alpha$-computable family such that $S$ contains at least two elements. Then the Rogers semilattice $R_{\Sigma^0_\alpha}(S)$ is infinite, and it is not a lattice.

Note that if $S$ contains precisely one element, then the structure $R_{\Sigma^0_\alpha}(S)$ is one-element.

Theorem (essentially, Dorzhieva 2016)

Let $n$ be a non-zero natural number. Let $S$ be a $\Pi^1_n$-computable family with at least two elements. Then the semilattice $R_{\Pi^1_n}(S)$ is infinite, and it is not a lattice.
Rogers semilattices for different levels

We say that an upper semilattice is *non-trivial* if it contains more than one element.

**Theorem (Podzorov 2008)**
Suppose that $m$ and $n$ are non-zero natural numbers with $m \geq n + 2$. If $\mathcal{U}$ is a non-trivial Rogers $\Sigma^0_m$-semilattice, then $\mathcal{U}$ is not isomorphic to any Rogers $\Sigma^0_n$-semilattice.

**Theorem (Badaev and Goncharov 2008)**
Suppose that $\alpha$ and $\beta$ are non-zero computable ordinals such that $\alpha \geq \beta + 3$. If $\mathcal{U}$ is a non-trivial Rogers $\Sigma^0_\alpha$-semilattice, then $\mathcal{U}$ is not isomorphic to any Rogers $\Sigma^0_\beta$-semilattice.
Rogers semilattices for different levels

Theorem 1 (B., Ospichev, and Yamaleev)
Suppose that $m$ and $n$ are non-zero natural numbers with $m \geq n + 1$. If $\mathcal{U}$ is a non-trivial Rogers $\Pi^1_m$-semilattice, then $\mathcal{U}$ is not isomorphic to any Rogers $\Pi^1_n$-semilattice.
Proof of Theorem 1: Outline

(1) First, we give an upper bound on the complexity of intervals inside Rogers $\Pi^1_n$-semilattices.

By $E_n$ we denote the $m$-complete $\Sigma^1_n$ set.

Lemma 1
Let $\mathcal{M}$ be a Rogers $\Pi^1_n$-semilattice. If $a <_\mathcal{M} b$ are elements from $\mathcal{M}$, then the interval $[a; b]_\mathcal{M}$, treated as a structure in the language $\{\leq, \lor\}$, has a $\Sigma^0_2(E_n)$ presentation.
Proof of Theorem 1: Outline

Recall that $m \geq n + 1$.

(2) We show that inside any non-trivial Rogers $\Pi^1_m$-semilattice, one can find a “sufficiently complicated” interval.

By applying the result of Downey and Knight (1992), one can choose a countable linear order $\mathcal{L}$ such that:

▶ $\mathcal{L}$ has the least and the greatest elements,
▶ $\mathcal{L}$ has a $\deg_T(E_n)^{(5)}$-computable copy, and
▶ $\mathcal{L}$ has no $\Sigma^0_2(E_n)$-computable presentations.

By Lemma 1, $\mathcal{L}$ cannot be realized as an interval inside a Rogers $\Pi^1_n$-semilattice.

On the other hand, $\mathcal{L}$ has a $\Delta^1_{n+1}$ presentation, and this is enough to obtain the following: Inside any non-trivial Rogers $\Pi^1_m$-semilattice, there exists an interval isomorphic to $\mathcal{L}$. 
Levels of the Ershov hierarchy

Let $n$ be a non-zero natural number. Recall that:

- A set $X \subseteq \mathbb{N}$ is $2n$-c.e. if there are c.e. sets $W_1, W_2, \ldots, W_{2n-1}, W_{2n}$ such that
  \[ X = (W_1 \setminus W_2) \cup (W_3 \setminus W_4) \cup \cdots \cup (W_{2n-1} \setminus W_{2n}). \]

- A set $X$ is $(2n + 1)$-c.e. if there are c.e. sets $W_1, W_2, \ldots, W_{2n-1}, W_{2n}, W_{2n+1}$ such that
  \[ X = (W_1 \setminus W_2) \cup (W_3 \setminus W_4) \cup \cdots \cup (W_{2n-1} \setminus W_{2n}) \cup W_{2n+1}. \]

By $\Sigma_m^{-1}$ we denote the class of all $m$-c.e. sets.
Levels of the Ershov hierarchy

The case of the Ershov hierarchy is quite different:

**Theorem (Herbert, Jain, Lempp, Mustafa, and Stephan)**

Let $m$ and $n$ be non-zero natural numbers with $m \geq n + 1$. Then every Rogers $\Sigma_n^{-1}$-semilattice is also a Rogers $\Sigma_m^{-1}$-semilattice.
Theorem (Ershov and Lavrov 1973)
There are finite families $F_i, i \in \omega$, of c.e. sets such that the Rogers semilattices $R_{\Sigma^0_1}(F_i), i \in \omega$, are pairwise non-isomorphic.

Theorem (V’yugin 1973)
There are $\Sigma^0_1$-computable families $S_i, i \in \omega$, such that the semilattices $R_{\Sigma^0_1}(S_i), i \in \omega$, are pairwise elementarily non-equivalent.
Number of isomorphism types

Theorem (Badaev, Goncharov, and Sorbi 2005)
Let $n \geq 2$ be a natural number. There are $\Sigma^0_n$-computable families $S_i, i \in \omega$, such that the semilattices $R_{\Sigma^0_n}(S_i), i \in \omega$, are pairwise elementarily non-equivalent.

Open Question 1
Let $n$ be a non-zero natural number. Are there infinitely many pairwise non-isomorphic Rogers $\Pi^1_n$-semilattices?
Elementary theories of Rogers $\Gamma$-semilattices

**Theorem (Badaev, Goncharov, Podzorov, and Sorbi 2003)**

Let $n \geq 2$ be a natural number. For any non-trivial Rogers $\Sigma^0_n$-semilattice $U$, its first-order theory $Th(U)$ is hereditarily undecidable.

**Theorem (Dorzhieva 2016)**

Let $n$ be a non-zero natural number. For any non-trivial Rogers $\Pi^1_n$-semilattice $U$, the theory $Th(U)$ is hereditarily undecidable.
Let $\mathcal{U}$ be a non-trivial Rogers $\Pi^1_n$-semilattice.

Dorzhieva (2016) provides a first-order interpretation (with parameters) of the poset $(\mathcal{E}^*; \subseteq^*)$ inside $\mathcal{U}$. An analysis of the interpretation shows the following:

**Corollary**
The set $\Pi_6-Th_{\leq}(\mathcal{U})$, i.e. the $\Pi_6$-fragment of the theory $Th(\mathcal{U})$ in the language $\{\leq\}$, is hereditarily undecidable.

The methods of the proof of Theorem 1 allow to obtain the following:

**Proposition**
- The fragment $\Sigma_1-Th_{\leq}(\mathcal{U})$ is decidable.
- The fragment $\Pi_3-Th_{\leq}(\mathcal{U})$ is hereditarily undecidable.
Problem 2
Study the complexity of the first-order theories for Rogers $\Gamma$-semilattices: What are their $m$-degrees?

In general, the problem seems to be quite hard to attack: e.g., recall that there are countably many elementary theories of Rogers $\Sigma^0_1$-semilattices [V’yugin 1973].

As a first step to tackle Problem 2, we introduce another approach.
The semilattice $\mathcal{R}$ of all numberings

Recall that in general, when we say that a numbering $\nu$ is reducible to a numbering $\mu$, we do not require that $\nu$ and $\mu$ are numberings of the same family. Indeed,

$$\begin{align*}
\nu &: \mathbb{N} \to \text{onto } S, \\
\mu &: \mathbb{N} \to \text{onto } T, \\
\nu &\leq \mu
\end{align*} \implies S \subseteq T.$$

Therefore, one can consider the following structure:

Let $Num := \{\nu : \nu$ is a function from $\mathbb{N}$ into $P(\mathbb{N})\}$. Then the quotient

$$\mathcal{R} := (Num; \leq, \oplus)/\equiv$$

is well-defined, and it is an upper semilattice. We say that $\mathcal{R}$ is the Rogers semilattice of all numberings.
One can also work with natural subsystems of $\mathcal{R}$:

Let $\Gamma$ be a complexity class (e.g., $\Sigma^0_2$ or $\Pi^1_1$). Then the structure

$$\mathcal{R}_\Gamma := (\{\nu \in Num : \nu \text{ is } \Gamma\text{-computable}\}; \leq, \oplus)/\equiv$$

is called the Rogers semilattice of all $\Gamma$-computable numberings.
The complexity of $Th(\mathcal{R}_{\Sigma_1^0})$

Theorem 2 (B., Mustafa, and Yamaleev 2019)

The theory $Th(\mathcal{R}_{\Sigma_1^0})$, i.e. the elementary theory of the Rogers semilattice of all $\Sigma_1^0$-computable numberings, is recursively isomorphic to first-order arithmetic. Moreover, the fragment $\Pi_5-\text{Th}(\mathcal{R}_{\Sigma_1^0})$ is hereditarily undecidable.
Proof of Theorem 2: Outline

(1) Recall the example from [Ershov 1977]: If $A \subsetneq B$ are c.e. sets, then the Rogers $\Sigma^0_1$-semilattice $R_{\Sigma^0_1}(\{A, B\})$ is isomorphic to the semilattice of c.e. $m$-degrees $R_m$.

This immediately gives a first-order interpretation (with parameters!) of $R_m$ inside $R_{\Sigma^0_1}$.

(2) In order to eliminate the parameters from the interpretation, we employ the following simple fact: Every minimal element inside $R_{\Sigma^0_1}$ is induced by a numbering of a one-element family. Thus, one can also provide a first-order definition for the elements induced by numberings of two-element families.

Recall that Nies (1994) proved that the theory $Th(R_m)$ is recursively isomorphic to first-order arithmetic. Since $R_m$ is first-order interpretable without parameters inside $R_{\Sigma^0_1}$, we deduce that $Th(R_{\Sigma^0_1}) \equiv_1 Th(R_m)$. 
The complexity of $Th(\mathcal{R})$

Recall that $\mathcal{R}$ is the semilattice of all numberings.

**Theorem 3 (B., Mustafa, and Yamaleev 2019)**

The theory $Th(\mathcal{R})$ is recursively isomorphic to second-order arithmetic. The fragment $\Pi_5$-$Th(\mathcal{R})$ is hereditarily undecidable.

The proof is similar to the previous one, modulo the following modification: We use that the elementary theory of the semilattice of all $m$-degrees $D_m$ is recursively isomorphic to second-order arithmetic [Nerode and Shore 1980].
The complexity of $Th(\mathcal{R}_{\Sigma^0_\alpha})$

We also obtain a partial result on the semilattice of all $\Sigma^0_\alpha$-computable numberings.

**Proposition (B., Mustafa, and Yamaleev 2019)**

Let $\alpha$ be a computable successor ordinal such that $\alpha \geq 2$. The elementary theory of the semilattice of $\Sigma^0_\alpha$ $m$-degrees is 1-reducible to $Th(\mathcal{R}_{\Sigma^0_\alpha})$. 
Open Question 2

Let $\alpha$ be a non-zero computable ordinal. Is it true that for any non-trivial Rogers $\Sigma^0_\alpha$-semilattices $\mathcal{U}$ and $\mathcal{V}$, the theories $Th(\mathcal{U})$ and $Th(\mathcal{V})$ are recursively isomorphic?

Note that the following question is still open:

Is it true that for any non-trivial Rogers $\Sigma^0_1$-semilattice $\mathcal{U}$, the theory $Th(\mathcal{U})$ is undecidable?
In the setting of analytical numberings, the following problem seems to be of interest:

**Open Question 3**

Study the connections between additional set-theoretic assumptions and the results on $\Pi^1_n$-computable numberings.

So far, there are only few results in this direction.
Friedberg numberings

A numbering \( \nu \) is Friedberg if for any natural numbers \( m \neq n \), we have \( \nu(m) \neq \nu(n) \).

**Theorem (Owings 1970)**
The family of all \( \Pi^1_1 \) sets has no \( \Pi^1_1 \)-computable Friedberg numbering.

**Theorem (Dorzhieva)**

(a) The family of all \( \Pi^1_2 \) sets has no \( \Pi^1_2 \)-computable Friedberg numbering.

(b) Assume \( V = L \). For any natural number \( n \geq 3 \), the family of all \( \Pi^1_n \) sets has no \( \Pi^1_n \)-computable Friedberg numbering.
II. Different kinds of reducibility between numberings:

(a) Relativized reducibility.
(b) $bm$-Reducibility.
Let $d$ be a Turing degree.

A numbering $\nu$ is $d$-reducible to a numbering $\mu$, denoted by $\nu \leq_d \mu$, if there is a $d$-computable function $f(x)$ such that $\nu(n) = \mu(f(n))$ for all $n$.

For a $\Gamma$-computable family $S$, one can introduce an upper semilattice

$$\mathcal{R}_d^\Gamma(S) := (\text{Com}_{\Gamma}(S); \leq_d, \oplus)/\equiv_d.$$
In the series of papers (2003), Badaev, Goncharov, Podzorov, and Sorbi obtained several results on semilattices \( R^{0(n)}_{\Sigma_0^m} (S) \), where \( n, m \in \omega \): in particular,

**Theorem (Badaev, Goncharov, and Sorbi 2003)**

Let \( m \) be a non-zero natural number. Consider the following two \( \Sigma^0_m \)-computable families:

- \( F \) is the family of all finite sets,
- \( S \) is the family of all \( \Sigma^0_m \) sets.

Then for any \( i < m \), the semilattices \( R^{0(i)}_{\Sigma_0^m} (F) \) and \( R^{0(i)}_{\Sigma_0^m} (S) \) are not isomorphic.
Theorem (Goncharov 1982)
Let $S$ be a $\Sigma^0_1$-computable family. Suppose that $S$ has two
$\Sigma^0_1$-computable Friedberg numberings $\nu$ and $\mu$ such that $\nu \not\equiv \mu$
and $\nu \equiv 0^\prime \mu$. Then $S$ has infinitely many pairwise non-equivalent,
$\Sigma^0_1$-computable Friedberg numberings.

This result has a counterpart in computable structure theory:

Theorem (Goncharov 1982)
Let $A$ be a computable structure. Suppose that $B$ is a computable
copy of $A$ such that $B \not\cong_{\Delta^0_1} A$ and $B \cong_{\Delta^0_2} A$. Then $A$ has infinite
computable dimension.
**bm-Reducibility**

Maslova (1979) introduced a *bounded version* of $m$-reducibility on sets:

Let $A$ and $B$ be subsets of $\mathbb{N}$. Then $A \leq_{bm} B$ if there is a total computable function $g(x)$ such that:

(a) $n \in A$ iff $g(n) \in B$, for all $n \in \mathbb{N}$;

(b) for any $k \in \mathbb{N}$, the preimage $g^{-1}(k)$ is a finite set.

Following her approach, we say that a numbering $\nu$ is **$bm$-reducible** to a numbering $\mu$, denoted by $\nu \leq_{bm} \mu$, if there is a total computable function $f(x)$ with the following properties:

1. $\nu(n) = \mu(f(n))$, for all $n \in \mathbb{N}$;

2. for any $k \in \mathbb{N}$, the set $f^{-1}(k)$ is finite.

Consider a $\Gamma$-computable family $S$. In a natural way, the reducibility $\leq_{bm}$ gives rise to the corresponding Rogers $bm$-semilattice:

$$\mathcal{R}^{bm}_\Gamma(S) := (Com_\Gamma(S); \leq_{bm}, \oplus)/\equiv_{bm}.$$
Cardinalities of $bm$-semilattices

Recall the result of Khutoretskii (1971):
For a $\Sigma^0_1$-computable family $S$, the semilattice $R_{\Sigma^0_1}(S)$ is either one-element, or infinite.

This result does not extend to the case of $\leq bm$:

Theorem 4 (B., Mustafa, and Ospichev)

- Let $S$ be a finite $\Sigma^0_1$-computable family with precisely $n \geq 2$ elements. Then the cardinality of the Rogers $bm$-semilattice $R^{bm}_{\Sigma^0_1}(S)$ is either equal to $2^n - 1$, or countably infinite. Both these cases can be realized.

- For any non-zero natural number $n$, there is an infinite $\Sigma^0_1$-computable family $T$ such that the cardinality of the $bm$-semilattice $R^{bm}_{\Sigma^0_1}(T)$ is equal to $2^n$. 
There is a $bm$-lattice

Recall the result of Selivanov (1976):
Let $S$ be a $\Sigma^0_1$-computable family. If the structure $R_{\Sigma^0_1}(S)$ is infinite, then it is not a lattice.

Again, this cannot be extended to the $bm$-case:

**Proposition (B., Mustafa, and Ospichev)**
Consider a $\Sigma^0_1$-computable family $S := \{\{k\} : k \in \mathbb{N}\}$. Then the structure $R_{\Sigma^0_1}^{bm}(S)$ is isomorphic to the lattice of all $\Pi^0_2$ sets (under inclusion).
Nevertheless, our results above do not transfer to further levels of the hyperarithmetical hierarchy:

Proposition (B., Mustafa, and Ospichev)
Let $\alpha \geq 2$ be a computable successor ordinal. Let $S$ be a $\Sigma^0_\alpha$-computable family such that $S$ contains at least two elements. Then the $bm$-semilattice $R^{bm}_{\Sigma^0_\alpha}(S)$ is infinite, and it is not a lattice.

Open Question 4
Provide a complete characterization for possible cardinalities of $bm$-semilattices $R^{bm}_{\Sigma^0_1}(S)$. 
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Isomorphism types of Rogers semilattices in the analytical hierarchy, submitted.

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