Computable dimension and projective planes

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- HKSS-completeness of pappian (desarguesian) projective planes
- Complexity of the computable categoricity problem

Computable structures and degree spectrum

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The *degree spectrum* of $\mathfrak{M}$ is the set

$$\mathrm{DgSp}(\mathfrak{M}) = \{\deg(\mathfrak{N}) \mid \mathfrak{N} \text{ is a presentation of } \mathfrak{M}\}.$$

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If R is a relation on the domain of a computably presentable structure $\mathfrak{M}$, then the *degree spectrum of R* on $\mathfrak{M}$ is the set

$$\mathrm{DgSp}_{\mathfrak{M}}(R) = \{\deg(f(R)) \mid f : \mathfrak{M} \to \mathfrak{N} \text{ is a computable presentation of } \mathfrak{M}\}.$$

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A *projective plane* is a structure $\mathfrak{A}=\langle A, A^0, {}^0A, \cdot\rangle$ with a disjunction of A into two subsets $A^0 \cup {}^0A = A$, $A^0 \cap {}^0A = \varnothing$ and commutative partial operation "$\cdot$" which satisfy the following properties:

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(1) $a{\cdot}b$ is defined iff a and b are distinct elements of the same type (a and b are of the *same type* if $a, b \in A^0$ or $a, b \in {}^0\!A$);

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(3) for all $a, b, c \in A$ if $a{\cdot}b$, $a{\cdot}c$, $(a{\cdot}b){\cdot}(a{\cdot}c)$ are defined, then $(a{\cdot}b){\cdot}(a{\cdot}c) = a$;

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(4) there exist distinct $a, b, c, d \in A$ such that products $a{\cdot}b$, $b{\cdot}c$, $c{\cdot}d$, $d{\cdot}a$ are defined and pairwise distinct ($\mathfrak{A}$ is *nondegenerate*.)

Familiar classes of projective planes

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$$\text{Freely Generated Planes} \cap \text{Desarguesian Planes} = \varnothing$$

Computable projective planes

In any projective plane $\mathfrak{A} = \langle A, A^0, {}^0A, \cdot \rangle$ we replace the binary operation by its graph

$$P^{\mathfrak{A}} = \{\langle a, b, c \rangle \in A^3 \mid a{\cdot}b \text{ is defined and } a{\cdot}b = c\},$$

and consider $\mathfrak{A}$ as a model of predicate signature

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and consider $\mathfrak{A}$ as a model of predicate signature

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So, $\mathfrak{A}$ is computable if $A, A^0, {}^0A$ are computable subsets of ω and $P^{\mathfrak{A}}$ is a computable subset of ω^3.

Definition of configuration

A *configuration* is a structure $\mathfrak{A}=\langle A, A^0, {}^0A, I\rangle$ with a disjunction of A into two subsets $A^0 \cup {}^0A = A$, $A^0 \cap {}^0A = \varnothing$ and symmetric binary relation $I \subseteq A^2$ (*incidence relation*) which satisfy the following properties:

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(2) if $\langle a, c\rangle \in I$, $\langle b, c\rangle \in I$, $\langle a, d\rangle \in I$, and $\langle b, d\rangle \in I$, then $a=b$ or $c=d$.

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Given a configuration $\mathfrak{A}$, we additionally define a partial operation as follows:

(3) a product $a{\cdot}b$ is defined and $a{\cdot}b = c$ iff $a \neq b$, $\langle a, c\rangle \in I$ and $\langle b, c\rangle \in I$.

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We can consider any projective plane $\mathfrak{A} = \langle A, A^0, {}^0A, \cdot\rangle$ as a configuration with the following incidence relation

$$I = \{\langle a, b\rangle \in A^2 \mid \exists x \in A(a{\cdot}x = b)\}.$$

Definition of free closure

We say that a configuration $\mathfrak{B}$ is a *free closure* of a configuration $\mathfrak{A}$ and write $\mathfrak{B} = \mathfrak{F}(\mathfrak{A})$, if there exists a countable sequence

$$\mathfrak{A} = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_i \subseteq \ldots$$

of configurations such that $\mathfrak{B} = \bigcup_{i \in \omega} \mathfrak{A}_i$ and for all $i \in \omega$ the following conditions hold:

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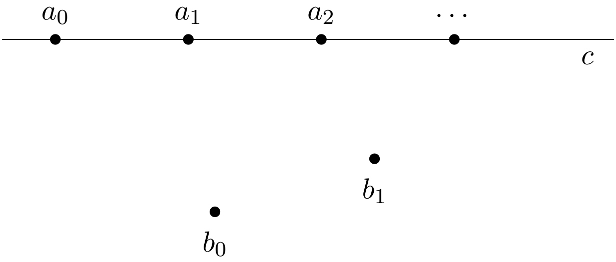
If $\mathfrak{F}(\mathfrak{A})$ is a projective plane, then we say that $\mathfrak{F}(\mathfrak{A})$ is *freely generated* by a configuration $\mathfrak{A}$.

Definition of free projective plane

The *free* projective plane $\mathfrak{F}_\alpha$, where $2 \leqslant \alpha \leqslant \omega$, is freely generated by the *standard* configuration, consisting of the set of points $\{b_0, b_1\} \cup \{a_i \mid i \in \alpha\}$, the singleton of lines $\{c\}$, and the incidence relation $\{\langle a_i, c\rangle, \langle c, a_i\rangle \mid i \in \alpha\}$.



Rank of free closure

If $\mathfrak{A} = \langle A, A^0, {}^0A, I \rangle$ is a finite configuration, then the *rank* of $\mathfrak{A}$ is the number

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The *rank* of a free closure $\mathfrak{F}(\mathfrak{A})$ is defined to be the rank of a configuration $\mathfrak{A}$.

Thus, the rank of free projective plane $\mathfrak{F}_n$, where $2 \leqslant n < \omega$, is the number $n + 6$. The rank of free projective plane $\mathfrak{F}_\omega$ is ω.

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Theorem 1. *Every countable free projective plane has computable dimension either* 1 *or* ω*. Furthermore, such a plane is computably categorical if and only if it has finite rank.*

Proof: We use the *unbounded models* method [Goncharov, 1980] to prove that the computable dimension of $\mathfrak{F}_\omega$ is effectively infinite.

Questions on freely generated projective planes

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Does there exist, for a given $n > 1$, a freely generated projective plane with computable dimension n?

Effective completeness of graphs

Theorem 2 [Hirschfeldt, Khoussainov, Shore, Slinko, 2002].
For every automorphically nontrivial, countable structure $\mathfrak{M}$ there exists a countable directed graph $\mathfrak{G}$ such that

(1) $\mathrm{DgSp}(\mathfrak{G}) = \mathrm{DgSp}(\mathfrak{M})$;

(2) *For any degree* $\mathbf{d}$, $\dim_{\mathbf{d}}(\mathfrak{G}) = \dim_{\mathbf{d}}(\mathfrak{M})$*;*

(3) *For any* $a \in |\mathfrak{M}|$ *there is an* $x \in |\mathfrak{G}|$ *such that* $\dim\langle\mathfrak{G}, x\rangle = \dim\langle\mathfrak{M}, a\rangle$*;*

(4) *For any* $R \subseteq |\mathfrak{M}|$ *there is a* $U \subseteq |\mathfrak{G}|$ *such that* $\mathrm{DgSp}_{\mathfrak{G}}(U) = \mathrm{DgSp}_{\mathfrak{M}}(R)$.

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Definition A class of structures K is *HKSS-complete*, if for every automorphically nontrivial, countable structure $\mathfrak{M}$ there exists a countable structure $\mathfrak{A} \in K$ such that $\mathfrak{A}$ satisfies properties (1)–(4) of Theorem 2.

Other HKSS-complete classes

Theorem 2 remains true if "directed graph" is replaced by a structure from any of the following classes:

- symmetric irreflexive graphs; (we will use this fact later)
- partial orderings;
- lattices;
- rings (with zero-divisors);
- integral domains (of arbitrary characteristic);
- commutative semigroups;
- 2-step nilpotent groups;
- fields (of zero characteristic). (we will use this fact later)

HKSS-completeness of freely generated projective planes

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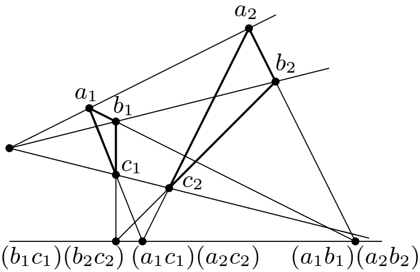
Corollary 4. *For every* $n \in \omega \cup \{\omega\}$ *there exists a freely generated projective plane of infinite rank with computable dimension* n.

Definition of desarguesian projective planes

A projective plane is *desarguesian* iff for each of its elements a_1, b_1, c_1, a_2, b_2, c_2 of the same type such that the products $a_1{\cdot}a_2$, $b_1{\cdot}b_2$, $c_1{\cdot}c_2$, $(a_1{\cdot}b_1){\cdot}(a_2{\cdot}b_2)$, $(a_1{\cdot}c_1){\cdot}(a_2{\cdot}c_2)$, $(b_1{\cdot}c_1){\cdot}(b_2{\cdot}c_2)$ are defined and the triples $\{a_1, b_1, c_1\}$, $\{a_2, b_2, c_2\}$ form nondegenerate triangles, if $a_1{\cdot}a_2$, $b_1{\cdot}b_2$, $c_1{\cdot}c_2$ are incident to the same element, then $(a_1{\cdot}b_1){\cdot}(a_2{\cdot}b_2)$, $(a_1{\cdot}c_1){\cdot}(a_2{\cdot}c_2)$, $(b_1{\cdot}c_1){\cdot}(b_2{\cdot}c_2)$ are also incident to the same element.



Definition of pappian projective planes

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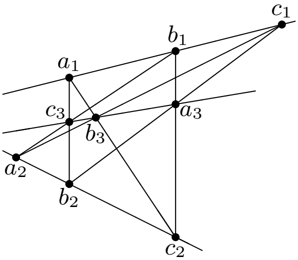
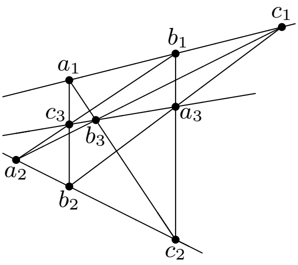


Definition of pappian projective planes

A projective plane is *pappian* iff for each of its elements a_1, b_1, c_1, a_2, b_2, c_2 of the same type such that $a_1{\cdot}b_1 = a_1{\cdot}c_1 = b_1{\cdot}c_1$, $a_2{\cdot}b_2 = a_2{\cdot}c_2 = b_2{\cdot}c_2$, $a_1{\cdot}b_1 \neq a_2{\cdot}b_2$, and the quadruple $\{a_1, b_1, a_2, b_2\}$ forms a nondegenerate quadrangle, if the products $a_3 = (b_1{\cdot}c_2){\cdot}(b_2{\cdot}c_1)$, $b_3 = (a_1{\cdot}c_2){\cdot}(a_2{\cdot}c_1)$, $c_3 = (a_1{\cdot}b_2){\cdot}(a_2{\cdot}b_1)$ are defined, then a_3, b_3, c_3 are incident to the same element.



Every pappian projective plane is desarguesian.

HKSS-completeness of pappian projective planes

Theorem 5 [Miller, Poonen, Schoutens, Shlapentokh, 2018].
The class of fields is HKSS-complete.

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Proof: We construct an effective coding (based on the idea of *coordinatization*) of fields into pappian projective planes preserving most computable-model-theoretic properties.

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Corollary 7. *For every* $n \in \omega \cup \{\omega\}$ *there exists a pappian (desarguesian) projective plane with computable dimension* n.

Definition of computable categoricity problem

If a structure $\mathfrak{M}$ is computable, then the *computable index* of $\mathfrak{M}$ is a number e such that $D(\mathfrak{M}) = W_e$, where W_e is a c.e. set with the Kleene number e.

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$$I(K) = \{e \in \omega \mid \mathfrak{M}_e \in K\}.$$

The *computable categoricity problem* for K is the set

$$I_{cc}(K) = \{e \in I(K) \mid \mathfrak{M}_e \text{ is computably categorical}\}.$$

Maximal complexity of $I_{cc}(K)$

If $I(K)$ is hyperarithmetical, then, at worst, $I_{cc}(K)$ is Π^1_1.

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Corollary 9. *The computable categoricity problem $I_{cc}(K)$ is m-complete Π^1_1 for each of the following classes K: symmetric irreflexive graphs, partial orderings, lattices, rings, commutative semigroups, fields, etc.*

Complexity of $I_{cc}(K)$ for projective planes

Theorem 10 *The computable categoricity problem for the class of free projective planes is m-complete Σ^0_3.*

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Theorem 11. *The computable categoricity problem $I_{cc}(K)$ is m-complete Π^1_1 for each of the following classes K:*

(1) *pappian projective planes,*
(2) *desarguesian projective planes,*
(3) *arbitrary projective planes.*

Complexity of $I_{cc}(K)$ for projective planes

Theorem 10 *The computable categoricity problem for the class of free projective planes is m-complete Σ_3^0.*

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Theorem 12 *The computable categoricity problem for the class of freely generated projective planes is m-complete Π_1^1 within the class.*