Definable sets in finite structures

Dugald Macpherson

University of Leeds

June 17, 2019

(joint work with Sylvy Anscombe, Charles Steinhorn, Daniel Wolf)

CDM Theorem

Theorem. [Chatzidakis, van den Dries and Macintyre 1992]
Let $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ be a formula in the language of rings. Then there is a positive constant C and finitely many pairs (d_i, μ_i) $(1 \leq i \leq K)$, with $d_i \in \{0, 1, \ldots, n\}$ and $\mu_i \in \mathbb{Q}^{>0}$ a positive rational number such that:
for each finite field $\mathbb{F}_q$, where q is a prime power, and each $\bar{a} \in \mathbb{F}_q^m$, if the set $\varphi(\mathbb{F}_q^n, \bar{a})$ is nonempty, then

$$\left| \left|\varphi\left(\mathbb{F}_q^n, \bar{a}\right)\right| - \mu_i q^{d_i} \right| < C q^{d_i - (1/2)}$$

for some $i \leq K$.
Moreover, for each pair (d_i, μ_i), there is a formula $\psi_i(y_1, \ldots, y_m)$ in the language of rings such that $\psi_i\left(\mathbb{F}_q^m\right)$ consists of those $\bar{a} \in \mathbb{F}_q^m$ for which the corresponding inequality with (d_i, μ_i) holds.

Proof uses Lang-Weil estimates + Ax's work on model theory of finite fields.

Asymptotic classes

The CDM theorem was turned into a **definition** of an abstract model-theoretic framework in work of M+Steinhorn, Elwes, Ryten,...

A **1-dimensional asymptotic class** is essentially a class of finite structures satisfying the conclusion of the theorem (e.g. class of finite fields).

Elwes (2007): notion of N-dimensional asymptotic class.

Ryten (PhD thesis, Leeds 2007): For any fixed Lie type τ, the class of all finite simple groups of type τ is an asymptotic class.

Asymptotic classes

So, e.g. the collection $\mathcal{C} = \{\mathrm{PSL}_2(q) : q \text{ prime power}\}$ is a 3-dim asymp. class; thus if $w(x_1, \ldots, x_n)$ is any word, we can let $\phi(\bar{x}, y)$ be the formula $w(x_1, \ldots, x_n) = y$ and get an asymptotic uniformity for the cardinalities of sets $\{\bar{x} \in G^n : w(\bar{x}) = a\}$ for $G \in \mathcal{C}$ and $a \in G$.

Pillay, Starchenko: Tao's 'Algebraic Regularity Lemma' holds for graphs uniformly definable in an asymptotic class.

Corresponding notion of **measurable** infinite structure (M + Steinhorn) – e.g. pseudofinite fields: measurable implies supersimple, finite SU rank

Fact: Any ultraproduct of an asymptotic class is measurable.

Goal: broaden this framework, e.g.

- allow parts of the structure (sorts? coordinatising geometries?) to vary independently,
- not require that ultraproducts have finite rank, or even have simple theory,
- not be specific about the form of the functions giving approximate cardinalities (no longer just of form μq^d).

Possible examples to keep in mind:

- Pairs $(V, \mathbb{F}_q)$ (2-sorted language), V a finite-dim vector space over $\mathbb{F}_q$;
- Disjoint unions of complete graphs all of same size (n copies of K_m);
- Finite abelian groups;
- Finite graphs of bounded degree.

Notation

Given class $\mathcal{C}$ of finite $\mathcal{L}$-structures and tuple $\bar{y}$ of variables, let $(\mathcal{C}, \bar{y})$ be the set

$$\left\{(M, \bar{a}) \;\middle|\; M \in \mathcal{C}, \bar{a} \in M^{|\bar{y}|}\right\}$$

of pairs consisting of a structure in $\mathcal{C}$ and a $\bar{y}$-tuple from that structure ('pointed structures in $\mathcal{C}$').

A finite partition Φ of $(\mathcal{C}, \bar{y})$ (i.e. partition into finitely many parts) is **$\emptyset$-definable** if for each $P \in \Phi$ there exists an $\mathcal{L}$-formula $\phi_P(\bar{y})$ without parameters such that for any $M \in \mathcal{C}$ and $\bar{a}$ in M, the following are equivalent:
(i) $(M, \bar{a}) \in P$
(ii) $M \models \phi_P(\bar{a})$.

Definition of R-m.a.c.

Let R be any set of functions $\mathcal{C} \longrightarrow \mathbb{R}_{\geq 0}$. A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is an ***R*-multidimensional asymptotic class** (***R*-m.a.c.**) if for every formula $\phi(\bar{x};\bar{y})$ there is a finite $\emptyset$-definable partition Φ of $(\mathcal{C},\bar{y})$ and a set $H_\Phi := \{h_P : P \in \Phi\} \subset R$ such that for each $P \in \Phi$,

$$\big| |\phi(\bar{x};\bar{a})| - h_P(M) \big| = o(h_P(M)) \tag{1}$$

for $(M,\bar{a}) \in P$ as $|M| \longrightarrow \infty$.

R-**m.e.c.** (multidimensional **exact** class) if above we have

$$|\phi(\bar{x},\bar{b})| = h_P(M).$$

Weak *R*-m.a.c. (or *R*-m.e.c.) – drop the definability clause on the partition Φ.

Observations

1. An asymptotic class is a m.a.c., where pairs (d, μ) are replaced by functions $M \mapsto \mu|M|^d$.

2. To prove a class $\mathcal{C}$ is an R-m.a.c. or R-m.e.c it suffices to work with formulas $\phi(x, \bar{y})$ (with x a single variable), replacing R by the ring generated by R. (Fibering argument, using definability.)

3. (Wolf) If $\mathcal{C}$ is a m.a.c. or m.e.c. then so is any class of finite structures uniformly bi-interpretable with $\mathcal{C}$. (Note: These conditions are not closed under uniform interpretability, as the definability clause may be lost.)

4. Any class uniformly interpretable in a m.a.c. is a weak m.a.c.; likewise for m.e.c.s.

5. Notion of **polynomial m.a.c** – here the functions in R are polynomials in the cardinalities of certain uniformly definable sets in M.

Examples of m.a.c.s

1. (Garcia, M, Steinhorn) Class $\mathcal{C}$ of 2-sorted structures $(V, \mathbb{F}_q)$, with V finite dim. v.s. over $\mathbb{F}_q$. Given $\phi(\bar{x}, \bar{y})$ there is a finite set E_ϕ of polynomials $g(\mathbb{V}, \mathbb{F})$ over $\mathbb{Q}$ such that if $M = (V, F)$ then each $h_P(M)$ has form $g(|V|, |F|)$ for some $g \in E_\phi$. Ultraproducts of $\mathcal{C}$ are supersimple, but the V-sort may have rank ω.

2. More generally, fix a quiver Q (digraph) of *finite representation type* (Dynkin diagrams A_n, D_n, E_q, E_7, E_8). Over a field F, this has a finite-dimensional *path algebra* FQ, which has finitely many isomorphism types of indecomposable representations. Let

$$\mathcal{C}_Q := \{(V, FQ, F) : F \text{ finite field }, V \text{ finite module for } FQ\}$$

(3-sorted, with the natural language). Then $\mathcal{C}_Q$ is a polynomial R-m.a.c. with the functions h_P given by polynomials $g(F, W_1, \ldots, W_t)$, where the W_i variables correspond to the indecomposables.

Examples

3. (Bello Aguirre) In the language of rings, for fixed $d \in \mathbb{N}$, let $\mathcal{C}_d$ be the collection of all finite residue rings $\mathbb{Z}/n\mathbb{Z}$, where n is a product of powers of at most d primes, each with exponent at most d.

Then $\mathcal{C}_d$ is a weak m.a.c., and a (polynomial) m.a.c. after appropriate expansion by unary predicates. If just one prime is involved, this is an asymptotic class. e.g. $\{\mathbb{Z}/p^2\mathbb{Z} : p \text{ prime}\}$ is a 2-dim asymptotic class. Ultraproducts are supersimple of finite SU-rank.
(Idea: $\mathbb{Z}/p^d\mathbb{Z}$ is coordinatised uniformly by $\mathbb{Z}/p\mathbb{Z}$.)

4. There is no weak m.e.c. consisting of infinitely many finite fields (consider elliptic curves).

Generalised measurable structures

Let $(S,+,\cdot,0,1,<)$ be a (commutative) ordered semiring (so $(S,+,0)$, $(S,\cdot,1)$ are commutative monoids, least element 0, etc.). Define $\sim$ on S with $a \sim b$ iff $a \leq b \leq na$ or $b \leq a \leq nb$ for some $n \in \mathbb{N}$. Put $D := S/\sim$, and $d : S \to D$ the natural 'dimension' map. Say S is a **measuring semiring** if

$$\forall x,y,z \in S((x < y \wedge d(y) = d(z)) \to x+z < y+z).$$

Let S be a measuring semiring and let M be an L-structure. We say that M is **S-measurable** if there is a function $h : \mathrm{Def}(M) \longrightarrow S$ such that

1. *finite sets* $h(X) = |X|$ for finite X;
2. *finite additivity* h is finitely additive;
3. *mac condition* for each $\emptyset$-definable family $\mathcal{X}$ there is finite $F \subseteq S$ such that $h(\mathcal{X}) = F$ and for each $f \in F$, $h^{-1}(f)$ is a $\emptyset$-definable family;
4. *Fubini* Suppose $p : X \longrightarrow Y$ is a definable function for which there exists $f \in S$ such that for all $\bar{a} \in Y$, $h(p^{-1}(\bar{a})) = f$; then we have $h(X) = f \cdot h(Y)$.

Generalised measurable structures

Weakly generalised measurable: in (3) above omit assumption that each $h^{-1}(f)$ is $\emptyset$-definable.

Proposition. Let M be weakly generalised measurable. Then

(i) M does not have the strict order property, i.e. there is no definable partial order with an infinite totally ordered subset;

(ii) M is **functionally unimodular**, that is, if $f_i : A \to B$ (for $i = 1, 2$) are definable surjections with f_i k_i-to-1, then $k_1 = k_2$.

Generalised measurable structures

Proposition. Let M be S-measurable, and let
$S_0 := \{h(X) : X \subseteq M \text{ definable.}\}$.
If $d(S_0) = S_0/\sim$ is well-ordered then M is supersimple.

Sketch Proof. Forking ensures drop in dimension.

Example (Anscombe). If M is a Fraïssé limit of a free amalgamation class then M is generalised measurable (note for example the generic triangle-free graph is such a Fraïssé limit and has TP1 and TP2 theory).

Generalised measurable structures

Proposition.
1. If $\mathcal{C}$ is a m.a.c. then any ultraproduct is generalised measurable (so NSOP, etc.)

2. If $\mathcal{C}$ is a m.e.c. then any ultraproduct is S-measurable for some ordered **ring** S.

Note. The above supersimplicity result applies to ultraproducts of examples like

$$\{(V, \mathbb{F}_q) : q \text{ prime power }, V \text{ finite dim. over } \mathbb{F}_q\}$$

and the quiver example, where the defining functions are given by polynomials in several variables, so the corresponding set of dimensions is well-ordered as they are polynomial degrees.

Examples of m.e.c.s

1. (Essentially by Pillay) Let M be any pseudofinite strongly minimal set. Then there is an R-m.e.c. $\mathcal{C}$ whose infinite ultraproducts are all elementarily equivalent to M, with the functions in R given as polynomials (over $\mathbb{Z}$) in the cardinalities of the members of $\mathcal{C}$.

2. (Wolf, based on Cherlin-Hrushovski) For a fixed language L and $d \in \mathbb{N}$, let $\mathcal{C}_{L,d}$ be the collection of all finite L-structures with at most d 4-types (i.e. orbits of the automorphism group on quadruples). Then $\mathcal{C}_{L,d}$ is a m.e.c (functions determining cardinalities are given by polynomials in the coordinatising Lie geometries).

3. For any fixed d, the class $\mathcal{C}_d$ of finite graphs of degree at most d is a m.e.c.

Examples of m.e.c.s – groups

4. (help from Kestner) The class of all finite abelian groups is a m.e.c. (in fact, for any fixed finite ring R, this holds for the class of all finite R-modules).

Proposition. If $\mathcal{C}$ is a m.e.c. of groups, then there is $d \in \mathbb{N}$ such that the groups in $\mathcal{C}$ have (uniformly definable) soluble radical $R(G)$ of index at most d, and $R(G)/F(G)$ has derived length at most d
(here $F(G)$ is the largest nilpotent normal subgroup of G).

Problem. Find a m.e.c. with an ultraproduct with non-simple theory.

Homogeneous structures as limits of m.e.c.s

Proposition. If M is a stable homogeneous structure (in Fraïssé's sense) over a finite relational language, then there is a m.e.c with ultraproduct elementarily equivalent to M.

Proof. Lachlan's structure theory for stable homogeneous structures + Wolf's result above.

Remark. Let q be a prime power with $q \equiv 1 \pmod 4$.
The **Paley graph** P_q has vertex set the field $\mathbb{F}_q$, with a adjacent to b if and only if $a - b$ is a square. The Paley graphs form a m.a.c (but not m.e.c) with limit the random graph, which is homogeneous but unstable.

Theorem. Let M be any of the following homogeneous structures. Then there is no m.e.c. with an ultraproduct elementarily equivalent to M.

(i) Any unstable homogeneous graph.

(ii) Any homogeneous tournament.

(iii) The digraph P_n for each $n \geq 3$ (universal subject to omitting an independent set I_n).

(iii) The generic bipartite graph.

Proof

of (ii) above, that the universal homogeneous tournament is not a limit of a m.e.c..

For a contradiction, consider a m.e.c. $\mathcal{C}$ of finite tournaments with ultraproduct $\equiv$ the random tournament.

1. Any finite regular tournament has indegree equal to outdegree, so has an odd number of vertices.

2. For any formula $\phi(\bar{x}, \bar{y})$, in a large enough finite tournament M the cardinality $|\phi(M, \bar{a})|$ depends just on the isomorphism type of $\bar{a}$ (uses QE, + definability clause of m.e.c.).

3. Any large enough $M \in \mathcal{C}$ is regular.

4. If $M \in \mathcal{C}$ is large enough finite and a, b are distinct vertices, then the tournaments M and on the sets $\{x : a, b \to x\}$, $\{x : x \to a, b\}$, $\{x : a \to x \to b\}$ and $\{x : b \to x \to a\}$ are all regular, so all of odd size.

5. The sum of four odd numbers +2 is even, contradicting (3)!