Structure of c-degrees of positive equivalences and preorders

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Main topics of the talk

Background. Computable reducibility.

Structure of degrees. Ceers of special kind.

Isomorphic types.

Definable sets of degrees.

Examples of positive equivalences and preorders

A c.e. equivalence (preorder) relation in ω is called positive equivalence (positive preoder).

$\mathcal{E}$ and $\mathcal{P}$ stand for the set of ceers and the set of positive preorders

Example 1. Words problem in algebra.

Example 2. Provable equivalence $\vdash \Phi \leftrightarrow \Psi$ and provable preorder $\vdash \Phi \rightarrow \Psi$ in the Peano arithmetics.
$\vdash_n$ is the restriction of $\vdash$ to Σ_n formulas.

Examples of positive equivalences and preorders

Example 3. Positive models.
A positive model is a countable algebraic structure $\mathcal{A} =< A, \sigma >$ equipped with a positive numbering $\nu : \omega \mapsto A$ s.t. predicates and functions of σ are represented by c.e. relations on ω and computable functions.
By definition, ν is a positive numbering if the numerical predicate $\nu(x) = \nu(y)$ is c.e. in x, y.

In particular, for $\sigma = \emptyset$, we have a natural example of positive equivalence, and, for $\sigma = \{\leq\}$, we have a positive preorder $P(x, y) \leftrightarrow \nu(x) \leq \nu(y)$.
Conversely, if $P(x, y)$ is a positive preorder then $\mathcal{A} =< \omega_{/\mathrm{supp}(P)}, \leq >$ with $\nu(x) = [x]_{\mathrm{supp}(P)}$ is a positive model.
Here $\mathrm{supp}(P)y \leftrightarrow xPy \,\&\, yPx$ is an equivalence relation that we call the support of P.

Examples of positive equivalences and preorders

Example 4. Splintors.
For a partial function f and any x, y, let
$xE_f y \leftrightarrow \exists m \exists n (f^{(m)}(x) \downarrow = f^{(n)}(y) \downarrow)$.

Yu.L. Ershov, 1971:
(i) Every ceer is E_f for some partial computable function f.
(ii) Every E_f is subobject of E_u where u is a unary universal partial computable function.

Example 5 of Gao and Gerdes.
E_R denotes the equivalence whose classes are singletons except one and the latter is R.

Computable reducibility

For any equivalences E_0, E_1, consider numberings $\nu_i : \omega \mapsto \omega_{/E_i}$ defined by: $\nu_i(x) = [x]_{E_i}$. Then, in the category of numbered sets, $(\omega_{/E_0}, \nu_0)$ is a subobject of $(\omega_{/E_1}, \nu_1)$ if there exist injective mapping $\mu : \omega_{/E_0} \mapsto \omega_{/E_1}$ and a computable function f s.t. for every $x \in \omega$, $\mu(\nu_0(x)) = \nu_1(f(x))$.

This means that $\forall x \forall y (\nu_0 x = \nu_0 y \Leftrightarrow \nu_1 f(x) = \nu_1 f(y))$ or, equivalently, $\forall x \forall y (x E_0 y \Leftrightarrow f(x) E_1 f(y))$.

Definition (C.Bernardi and A.Sorbi, 1983) with reference to numbering theory. A ceer E is called c-reducible to a ceer G ($E \leqslant_c G$) if there is a computable f s.t. $\forall x \forall y (x E y \Leftrightarrow f(x) G f(y))$. (For instance, in Example 4 $E_f \leqslant_c E_u$.)

Observation. The definition is easily extended without any change to c-reduction for binary relations, and, in particular, for positive preorders.

Structures of c-degrees

E is c-equivalent to G ($E \equiv_c G$) if $E \leqslant_c G$ and $G \leqslant_c E$.
$\deg(E) = \{G : G \equiv_c E\}$ is the degree of E.
$\mathcal{E}_c \leftrightharpoons < \{\deg(E) : E \in \mathcal{E}\}, \leqslant_c >$ is a partially ordered set.
$\mathcal{P}_c$ is defined analogously.

Proposition. Mapping $R \mapsto E_R$ induces isomorphism of the structure of 1-degrees of the infinite c.e. sets onto some interval of $\mathcal{E}_c$.

Due to A.Lachlan, 1969, and other well-known facts, this implies that:
elementary theory of $\mathcal{E}_c$ is undecidable;
$\mathcal{E}_c$ is not neither upper nor low semilattice;
$\mathcal{E}_c$ contains infinite chains of type ω;
$\mathcal{E}_c$ contains infinite anti-chains, etc.

General facts on $\mathcal{E}_c$

Let Id_n and Id stand for the equivalence modulo n and the identity relation.

Fact 1. $\mathrm{Id}_n \leqslant_c \mathrm{Id}_m \iff n \leq m$.
Fact 2. Any ceer with finitely many classes is c-equivalent to one of Id_n.
Fact 3. Every Id_n is c-reducible to any ceer with infinitely many classes. The converse is always wrong.
Fact 4. Id is not c-reducible to E_R with a simple set R (Example 5).
Fact 5. $\mathcal{E}_c$ has the least degree and the greatest degree. Any element of the latter is called universal positive equivalence.

Universal ceers.

Universal ceers do exist:
Yu.L. Ershov, 1971: E_u;
C.Bernardi and A.Sorbi, 1983: direct sum of all ceers, $\vdash_n$;
F.Montagna, 1982: $\vdash$, $u.f.p.$, e-complete;
U.Andrews, S.Lempp, et al, 2014: uniformly effectively inseparable.

Note: E_u is computably isomorphic to every $\vdash_n$ but not to $\vdash$.

Problem: How many isomorphic types there is inside a given c-degree?

S.Badaev and A.Sorbi, 2016: There is infinite anti-chain of non-isomorphic universal ceers.

Universal positive preorders do exist too:
F.Montagna and A.Sorbi, 1985: based on proof theory,
S.Badaev, B.Kalmurzayev, et al, 2018: based on direct sums.

Precompleteness and its variations.

A.I.Mal'tsev, 1961: An equivalence E is called precomplete if every partial computable function f could be totalized by a computable function g modulo E:
$\forall x(f(x)\downarrow\Rightarrow g(x)Ef(x))$.

F.Montagna, 1982: Non-trivial equivalence E is called uniformly finitely precomplete (u.f.p) if there is a computable function $f(D, e, x)$ s.t. for every finite set D and every e, x,
$\varphi_e(x)\downarrow[D]_E \Rightarrow \varphi_e(x)Ef(D, e, x)$.

Examples of u.f.p ceers: $\vdash$, $\vdash_n$, precomplete ceers.

F.Montagna, 1982: Every u.f.p. ceer is universal.

Precompleteness and its variations.

Ershov's fixed point theorem:
An equivalence E is precomplete iff there is a computable function f s.t.
$\forall e(\varphi_e(f(e))\downarrow\Rightarrow f(e)E\varphi_e(f(e)))$.

S.Badaev, 1991: An equivalence E is called weakly precomplete if there is a partial computable function f s.t.
$\forall e(\varphi_e \text{ is total } \Rightarrow f(e)E\varphi_e(f(e)))$.

S.Badaev and A.Sorbi, 2016: A ceer E is weakly precomplete iff
$\forall e(\varphi_e \text{ is total } \Rightarrow \exists n(nE\varphi_e(n))$.

S.Badaev and A.Sorbi, 2016: There is infinitely many computably non-isomorphic weakly precomplete universal ceers.

Complete information on universal ceers is in the survey of U.Andrews, S.Badaev, and A.Sorbi, 2017.

Jump operator. Criterion of universality

Definition (S.Gao and P.Gerdes, 2001):
$xE'y \Leftrightarrow x = y \vee \varphi_x(x)\downarrow E\varphi_y(y)\downarrow$.

1. $E \leqslant_c E'$.
2. $E_1 \leqslant_c E_2 \Leftrightarrow E_1' \leqslant_c E_2'$.
3. If E is not universal then E' is not universal.
4. U.Andrews and A.Sorbi, 2018: For every $E \in \mathcal{E}$, E' is uniform join-irreducible.

U.Andrews, S.Lempp, J.S.Miller, K.M.Ng, L.San Mauro, and A.Sorbi, 2014: Ceer E is universal iff $E' \equiv_c E$.

Some classes of ceers.

$\mathcal{F} \risingdotseq \{$degrees of ceers with finitely many classes$\}$,
$\mathcal{L} \risingdotseq \{$degrees above $\deg(\mathrm{Id})\}$ (each consists of light ceers),
$\mathcal{D}$ is the set of remaining degrees (each consists of dark ceers).

Properties of dark ceers

1. Every dark ceer has infinitely many equivalence classes and is incomparable with Id.
2. $\mathcal{D}$ is an ideal of $\mathcal{E}_c$ over $\mathcal{F}$.
3. $\mathcal{D}$ is closed under uniform join $\oplus$.
Here, $x(E_0 \oplus E_1)y \leftrightarrow 2xE_0 2y \& (2x+1)E_1(2y+1)$.
4. $\forall D \in \mathcal{D}(D \oplus \mathrm{Id} = D \vee \mathrm{Id})$.
5. $\mathcal{D}$ has neither maximal nor greatest elements. Any two incomparable elements has no join. (U.Andrews and A.Sorbi, arXiv).
6. $\mathcal{D}$ has infinitely many minimal elements over $\mathcal{F}$ induced by weakly precomplete ceers (U.Andrews and S.Badaev, to appear).

One more class of ceers

Observation. For every ceer E, either $E \equiv_c E \oplus \mathrm{Id}_1$ or $E <_c E \oplus \mathrm{Id}_1$.

Definition (A.Sorbi, folklor). E is called self-full if $E \equiv_c E \oplus \mathrm{Id}_1$ or, equivalently, if any computable reduction of E to itself is onto on equivalence classes.

Let $\mathcal{S}$ stands for the set of the degrees of self-full ceers. Then $\mathcal{D} \subset \mathcal{S}$.

U.Andrews and A.Sorbi, arXiv: If E is self-full then $\deg(E)$ has the unique strong minimal cover (the degree of $E \oplus \mathrm{Id}_1$), otherwise, $\deg(E)$ has infinitely many strong minimal covers.

Number of types of computable isomorphism

For a ceer E, $N(E)$ stands for the number of isomorphism types inside $\deg(E)$.

U.Andrews and S.Badaev, to appear: $N(E) = 1$ iff E is self-full and has no computable classes. Otherwise, $N(E) = \omega$.

For instance, a dark weakly precomplete ceer E has $N(E) = 1$.

D.Kabylzhanova, 2018: For a positive preorder P, $N(P) = 1$ iff P is self-full and has no computable classes. Otherwise, $N(P) = \omega$.

$N^\star(E)$ stands for the number of isomorphism types of weakly precomplete ceers $\deg(E)$.
U.Andrews and S.Badaev, to appear: The range of $N^\star(E)$ is $[0, \omega]$.

Definable sets of degrees

U.Andrews and A.Sorbi, arXiv:$deg_c(Id)$, $\mathcal{F}$, $\mathcal{L}$, $\mathcal{D}$, and $\mathcal{S}$ are first order definable.

Thank you for attention!

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