On sets and functions definable in ordered Abelian groups

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Model theory of pure ordered groups

- Presburger arithmetics.
- Regular ordered groups.
- Quantifier elimination for pure ordered groups.

Presburger arithmetic

Let $\Sigma = \{=, +, 0, 1\}$.

The axioms are the universal closures of the following:

1. $\neg(0 = x + 1)$;
2. $x + 1 = y + 1 \to x = y$;
3. $x + 0 = x$;
4. $x + (y + 1) = (x + y) + 1$;
5. $(P(0) \wedge \forall x(P(x) \to P(x + 1))) \to \forall y P(y)$, whenever P is a Σ-formula.

Theorem (Presburger, 1929)

Presburger arithmetic is consistent, complete, and decidable.

M. Presburger (1930). "Über der Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchen die Addition als einzige Operation hervortritt". In F. Leja (ed.). Comptes Rendus Premier Congrs̀ des Mathématicienes des Pays Slaves, Varsovie 1929 / Sprawozdanie z I Kongresu matematyków krajów slowiańskich, Warszawa 1929. Warsaw, Lwów and Krakow. pp. 92–101, 395.

Regular groups

Theorem (A. Robinson, E. Zakon, 1960)

All regularly discrete groups are elementarily equivalent. Two regularly dense groups, A and B, are elementarily equivalent if, and only if, they have the same prime invariants, i.e. if $[p]A = [p]B$ for every prime p.

Abraham Robinson and Elias Zakon, Elementary properties of ordered Abelian groups, Trans. AMS, 96 (1960), 222–236.

Theorem (E. Zakon, 1961)

All archimedean groups are regularly ordered.

Elias Zakon, Generalized archimedean groups, Trans. AMS, 99 (1961), 21–40.

O. Belegradek, Poly-regular ordered abelian groups, in: Yi Zhang (Ed.), Logic and Algebra (Contemp. Math. 302), AMS, Providence, RI, 2003.

Quantifier elimination for ordered Abelian groups

Theorem (Y. Gurevich, P. Schmitt, 1984)

The theory of ordered Abelian groups does not have the independence property.

Y. Gurevich, Elementary properties of ordered Abelian groups, Am. Math. Soc., Transl., II. Ser. 46 (1964) 165–192 (English, Russian original).

Y. Gurevich and P. H. Schmitt, The theory of ordered abelian groups does not have the independence property, Trans. Amer. Math. Soc. 284 (1984) 171–182.

This theorem has been rediscovered by R. Cluckers, I. Halupczok in 2011.

R. Cluckers, I. Halupczok, Quantifier elimination in ordered abelian groups, Confluentes Mathematici, Vol. 3, No 4 (2011) 587–615.

Quantifier elimination for some ordered Abelian groups

V. Weispfenning, Elimination of quantifiers for certain ordered and lattice-ordered abelian groups, Proc. of the Model Theory Meeting (Univ. Brussels, Brussels/Univ. Mons, Mons, 1980), Vol. 33 (1981) 131–155.

Ibuka Shingo; Kikyo Hirotaka; Tanaka Hiroshi. Quantifier Elimination for Lexicographic Products of Ordered Abelian Groups. Tsukuba J. Math. 33 (2009), 95–129.

Theorem (F. Rafel, 2017)

Let G be an ordered Abelian group with bounded regular rank. Then G admits QE in the following language:

$$\begin{aligned} L = &\{+,-,0,\leq\} \cup \{1_\Delta \mid \Delta \in \mathrm{RJ}(G),\ G/\Delta \ \textit{discrete}\} \cup \\ &\{x \equiv y \mod \Delta \mid \Delta \in \mathrm{RJ}(G)\} \cup \\ &\{x \equiv y \mod (\Delta + p^m G) \mid p \in \mathbb{P},\ \Delta \in \mathrm{RJ}_p(G),\ m \geq 1\} \end{aligned}$$

Farré, Rafel. (2017). Strong ordered Abelian groups and dp-rank, preprint.

Classes of ordered groups

- o-minimal;
- weakly o-minimal;
- quasi-o-minimal (coset-minimal);
- weakly quasi-o-minimal;
- eventually coset-minimal and periodic coset-minimal;
- dp-minimal;
- o-stable;
- ...

All of these groups are Abelian.

O-minimal ordered groups

Theorem (A. Pillay, C. Steinhorn)

Any o-minimal ordered groups is Abelian and divisible. That is in the language of pure ordered groups they are elementary equivalent to $(\mathbb{Q}, <, +, 0)$.

Theorem (A. Pillay, C. Steinhorn)

Any o-minimal ordered field is real closed.

A. Pillay, C. Steinhorn, Definable sets in ordered structures.1 Trans. Amer. Math. Soc. 295 (1986) 565–592.

Sorry, I omit details of o-minimality, it is too wide.

Weakly o-minimal ordered groups

Definition (M. Dickmann, D. Macpherson, D. Marker, C. Steinhorn)

An ordered sturcture is said to be weakly o-minimal if any definable subset is a finite union of convex subsets.
A theory is weakly o-minimal if all its models are.

Theorem (D. Macpherson, D. Marker, C. Steinhorn)

Any weakly o-minimal ordered group is Abelian and divisible. That is in the language of pure ordered groups they are elementary equivalent to $(\mathbb{Q},<,+,0)$.

Theorem (D. Macpherson, D. Marker, C. Steinhorn)

Any weakly o-minimal ordered field is real closed.

D. Macpherson, D. Marker, C. Steinhorn, Weakly o-minimal structures and real closed fields, Trans. Amer. Math. Soc. 352 (2000) 5435–5483.

Expansions of weakly o-minimal ordered groups

Problem by D. Macpherson, D. Marker, C. Steinhorn:
Does any weakly o-minimal ordered group (in an expanded language) have a weakly o-minimal theory?

Theorem (V. Verbovskiy, 2001)

There exists an example of a weakly o-minimal ordered group whose elementary theory is not weakly o-minimal.

Theorem (R. Wencel, 2008)

Any weakly o-minimal non-valuational ordered group has a weakly o-minimal theory. Moreover, the topological closure for definable cuts with the full induced structure is o-minimal.

R. Wencel, Weakly o-minimal non-valuational structures, Annals of Pure and Applied Logic, 154 (2008) 139–162.

Quasi-o-minimal ordered groups

Definition (O. Belegradek, et al.)

An ordered structure is said to be quasi-o-minimal if any its definable subset is a Boolean combinatrion of intervals and $\emptyset$-definable subsets.

Theorem (O. Belegradek, et al.)

Quasi-o-minimal ordered groups are abelian; they, as a pure ordered group, are an ordered abelian group of finite regular rank.

O. Belegradek, Y. Peterzil, and F. Wagner, Quasi-o-minimal structures, J. Symbolic Logic 65:3 (2000) 1115–1132.
O. V. Belegradek, A. P. Stolboushkin, M. A. Taitslin, Extended order-generic queries, Ann. Pure Appl. Logic 97 (1999) 85–125.

Coset-minimal ordered groups

Definition (F. Point, F. Wagner, 2000)

An ordered group is said to be coset-minimal if any its definable subset is a union of intervals intersected with cosets of definable subgroups.

Theorem (O. Belegradek, V. Verbovskiy, F. Wagner, 2003)

Pure coset-minimal and eventually coset-minimal groups are classified. In a discrete coset-minimal group every definable unary function is piece-wise linear. A dense coset-minimal group has the exchange property (which is false in the discrete case; moreover, any definable unary function is piecewise linear, except possibly for finitely many cosets of the smallest definable convex nonzero subgroup.

F. Point and F. O. Wagner, Essentially periodic ordered groups, Ann. Pure Appl. Logic 105:(1-3) (2000) 261-291.
O. Belegradek, V. Verbovskiy, F. Wagner, Coset-Minimal Groups, Annals of Pure and Applied Logic, 121 (2003) 113–143.

Weakly quasi-o-minimal ordered groups

Theorem (Kudaibergenov, 2010)

Every weakly quasi-o-minimal theory lacks the independence property

Theorem (Kudaibergenov, 2010)

Every weakly quasi-o-minimal ordered group is Abelian, every divisible Archimedean weakly quasi-o-minimal ordered group is weakly o-minimal, and every weakly o-minimal quasi-o-minimal ordered group is o-minimal.

Theorem (Kudaibergenov, 2010)

Every weakly quasi-o-minimal Archimedean ordered ring with nonzero multiplication is a real closed field that is embeddable into the field of reals.

K. Zh. Kudaibergenov, Weakly quasi-o-minimal models, Siberian Adv. Math., 20:4 (2010), 285–292.

New classes of ordered groups

- The definitions of the previous classes are based on description of definable subsets.
- Below we consider dp-minimal and o-stable ordered groups, whose definitions are based on properties of uniformly definable families of definable subsets.
- So, one of the first problem for these groups is to describe definable subsets as well as classification of these ordered groups.

Dp-minimality

Definition (Shelah)

An independence (or inp-) pattern of length κ is a sequence of pairs$(\phi^\alpha(x, y), k^\alpha)_{\alpha<\kappa}$ of formulas such that there exists an array $\langle a_i^\alpha : \alpha < \kappa, i < \lambda \rangle$ for some $\lambda \geq \omega$ such that :

- Rows are k^α-inconsistent : for each $\alpha < \kappa$, the set $\{\phi^\alpha(x, a_i^\alpha) : i < \lambda\}$ is k^α-inconsistent,
- paths are consistent : for all $\eta \in \lambda^\kappa$, the set $\{\phi^\alpha(x, a_{\eta(\alpha)}^\alpha) : \alpha < \kappa\}$ is consistent.

Definition

- (Goodrick) A theory is inp-minimal if there is no inp-pattern of length two in a single free variable x.
- (Onshuus and Usvyatsov) A theory is dp-minimal if it is *NIP* and inp-minimal.

Dp-minimal ordered groups

Theorem (P. Simon, 2011)

The linearly ordered dp-minimal ordered groups are abelian.

Theorem (P. Simon, 2011)

An infinite definable subset in a dp-minimal ordered divisible group has non-empty interior.

Simon, Pierre. On dp-minimal ordered structures. J. Symbolic Logic 76 (2011), no. 2, 448–460.

Dp-minimal ordered groups

An ordered group Γ is non-singular if $\Gamma/p\Gamma$ is finite for all primes p.

Theorem (F. Jahnke, P. Simon, E. Walsberg)

The following are equivalent:

1. $(\Gamma, +, \leq)$ *is non-singular.*
2. $(\Gamma, +, \leq)$ *is dp-minimal.*
3. *There is a definitional expansion of* $(\Gamma, +, \leq)$ *by countably many formulas which is weakly quasi-o-minimal.*

F. Jahnke, P. Simon, E. Walsberg, Dp-minimal valued fields. The Journal of Symbolic Logic, 82(1) (2017) 151–165.

Dp-minimal ordered field

Given a field k and an ordered abelian group Γ, $k((t^\Gamma))$ is the field of Hahn series with coefficients in k and exponents in Γ. By the Ax-Kochen/Ersov Theorem a field K admitting a henselian valuation with residue characteristic zero, residue field k and value group Γ is elementarily equivalent to $k((t^\Gamma))$.

Theorem (F. Jahnke, P. Simon, E. Walsberg)

The ordered field F is dp-minimal if and only if $F \equiv \mathbb{R}((t^\Gamma))$ for some non-singular ordered abelian group Γ.

F. Jahnke, P. Simon, E. Walsberg, Dp-minimal valued fields. The Journal of Symbolic Logic, 82(1) (2017) 151–165.

O-stable theories: preliminaries

A partition $\langle C, D\rangle$ of an ordered structure M is called a *cut* if $C < D$.

Given a cut $\langle C, D\rangle$ one can construct a partial type $\{c < x < d : c \in C,\ d \in D\}$.

Let s be a partial 1-type, A a set. Then

$$S^1_s(A) := \{p \in S^1(A) : p \cup s \text{ is consistent}\}.$$

Note, s need not to be a partial type over the set A.

O-stable theories: definition

Definition

1. An ordered structure $\mathcal{M}$ is *o-stable in* λ if for any $A \subseteq M$ with $|A| \leq \lambda$ and for any cut $\langle C, D\rangle$ in $\mathcal{M}$ there are at most λ 1-types over A which are consistent with the cut $\langle C, D\rangle$, i.e.
$$\left|S^1_{\langle C,D\rangle}(A)\right| \leq \lambda.$$
2. A theory T is *o-stable in* λ if every model of T is. Sometimes we write T is o-λ-stable.
3. A theory T is *o-stable* if there exists an infinite cardinal λ in which T is o-stable.
4. A theory T is *o-superstable* if there exists a cardinal λ such that T is o-stable in all $\mu \geq \lambda$.
5. A theory T is *strongly o-stable* if in addition to its o-stability any definable cut in any model $\mathcal{M}$ of T is definable in the language of pure ordering, or, equivalently, if $\sup A \in M$ for any definable subset A of $\mathcal{M}$.

O-stable theories: basic facts

Theorem (B. Baizhanov, V. Verbovskiy, 2011)

O-stable theories are NIP. O-minimal theories are strictly o-ω-stable, weakly o-minimal theories are o-ω- stable, and weakly quasi-o-minimal theories are o-superstable.

B. Baizhanov, V. Verbovskiy, O-stable theories, Algebra and Logic, 50:3 (2011), 211–225.

Theorem (V. Verbovskiy, 2010)

Dp-minimal theories with a linear order are o-stable.

A simple corollary of P. Simon. On dp-minimal ordered structures. J. Symbolic Logic 76 (2011), no. 2, 448–460.

Theorem

Dp-minimal theories with a linear order are o-superstable.

O-stable groups: basic facts

Theorem (V. Verbovskiy, 2010)

Any o-stable group is Abelian

V.V. Verbovskiy. O-stable ordered groups. Siberian Advances in Mathematics, 22 (2012), 50–74.

Theorem (V. Verbovskiy, 2010)

A pure ordered group whose elementary theory is o-ω-stable is divisible.

O-superstable groups: basic facts

Lemma (V. Verbovskiy, 2010)

If the elementary theory of G is o-superstable, then $|G : nG| < \infty$ for any positive integer n.

By F. Jahnke, P. Simon, E. Walsberg (2017) if $|G : nG| < \infty$ for any positive integer n, then G is dp-minimal. Then it is o-superstable.

Theorem

Let G be a pure ordered group. The elementary theory of G is o-superstable if and only if $|G : nG| < \infty$ for any positive integer n.

Theorem

Let G be a pure superstable orderable group. Then expansion of G by any order, which makes this group ordered, has o-superstable theory.

O-stable groups: basic facts

Theorem

Let G be a pure ordered group. The elementary theory of G is o-stable if and only if it is Abelian and for each n the formula $\Phi(x; y) = \exists t(0 \leq t \leq y \wedge n \mid (y - t))$ has no strict order property inside the cut $+\infty$.

Theorem

There exists be a pure stable orderable group G, such that there are two expansions of G by a linear order, each of which makes this group ordered, and one expansion has o-stable theory, while the other not.

Some definitions

For any definable convex set A each of its boundaries $\inf A$ and $\sup A$ defines convex subgroups H_A^- and H_A^+, respectively, in the following way:

$$H_A^- := \{g \in G : a - |g| \in A \text{ for any } a \in A\},$$
$$H_A^+ := \{g \in G : a + |g| \in A \text{ for any } a \in A\}.$$

We call H_A^- the left shore of A and H_A^+ the right shore of A.

The eventual stabilizer

$$K_\varphi(x) := \exists z \forall y \left[z < y \to (\varphi(y) \leftrightarrow \varphi(y + x))\right]$$

of a formula φ, which is consistent with the cut $+\infty$.

Eventual stabilizer

Theorem (V. Verbovskiy, 2010)

Let G be an o-stable ordered groups. If a formula is eventually minimal at the cut $+\infty$, then its eventual stabilizer is unbounded.

Theorem (V. Verbovskiy, 2010)

Let G be an o-stable ordered groups. For any formula which is consistent with the cut $+\infty$ there exists a subformula which is eventually minimal at the cut $+\infty$.

Theorem (V. Verbovskiy, 2010)

There exists an o-stable ordered group, which has an unbounded definable subset, whose eventual stabilizer is the whole group. Any such kind of a group has unboundedly many definable convex subgroups.

Definable subsets of o-stable ordered fields

Let G be an ordered groups.

If G is dp-minimal, then any infinite definable subset has a non-empty interior.

This is no more true for o-stable groups, because $(\mathbb{R}, <, +, 0, Q)$ is o-stable, where Q names the set of rational numbers.

Theorem (V. Verbovskiy, 2019)

Let F be an o-stable ordered field. Then any infinite definable subset has a non-empty interior.
In particular, any definable subgroup of the additive group is convex (V. Verbovskiy, 2010).

The number of convex subgroups

We say that G contains *boundedly many definable convex subgroups* if there is a cardinal λ, such that in any group which is elementary equivalent to G the number of convex definable subgroups does not exceed λ.
Otherwise we say that G has *unboundedly many definable convex subgroups*.

Theorem (Verbovskiy, 2010)

Let F be an o-stable ordered field. If its additive group has boundedly many definable convex subgroups, then F is weakly o-minimal and, as a corollary, real closed.

Remark. For an ordered field to have boundedly many definable convex subgroups of the additive group is equivalent to that the additive group has no non-trivial definable convex subgroups.

Conjecture. Any weakly o-minimal expansion of an ordered groups has a weakly o-minimal theory if and only if this expansion has boundedly many definable convex subgroups.

When a group has boundedly many definable convex subgroups

Let B be a definable subset.
Let E be the equivalence relation whose classes are convex components of B or of the complement of B.
Then B generates the family of definable convex subgroups in the following way:

$$\mathcal{F}_B = \{H^-_{[a]} : a \in G\} \cup \{H^+_{[a]} : a \in G\},$$

where $[a]$ is the E-class contained a.

Theorem

An ordered group has boundedly many definable convex subgroup if and only if $\mathcal{F}_B$ is finite for any definable subset B of any elementary extension of this group.

Also we define the following notations:

$$H^0_B = \bigcap\{H \in \mathcal{F}_b : H \neq \{0\}\}, \quad H^\infty_B = \bigcup\{H \in \mathcal{F}_b : H \neq G\}$$

Definable subsets of dp-minimal ordered groups

Let P be some property and $A \subseteq G$. We say that the property P holds *eventually in A* if there is an element $a \in G$ such that $a < \sup A$ and the property P holds on the intersection $(a, \infty) \cap A$. If $A = G$, we simply write that the property P *holds eventually*.

$$H_B^0 = \bigcap \{H \in \mathcal{F}_b : H \neq \{0\}\}, \quad H_B^\infty = \bigcup \{H \in \mathcal{F}_b : H \neq G\}$$

Lemma (J. Goodrick, V. Verbovskiy, 2019)

Let G be an ordered group whose elementary theory is dp-minimal, and let B be a definable unbounded subset of G such that $H_B^\infty \neq G$ as well as $H^\infty_{(B+g)\cap nG, nG} \neq nG$, whenever $g \in G$ and n is a positive integer. Then eventually B is a finite union of cosets of nG, for some n.

Definition

A definable subset B of an ordered group G is said to be of *finite valuation* if $\mathcal{F}_{(B+g)\cap G_1, G_1}$ is finite for any definable subgroup G_1 of G and any element g.

Theorem (J. Goodrick, V. Verbovskiy, 2019)

Let G be an ordered group whose elementary theory is dp-minimal, and let B be its definable subset of finite valuation. Then B is a finite union of convex sets intersected with cosets of definable subgroups.

Corollary (J. Goodrick, V. Verbovskiy, 2019)

Let G be an ordered group whose elementary theory is dp-minimal, and let G has boundedly many definable convex subgroups. Then any definable subset is a finite union of convex sets intersected with cosets of definable subgroups of the form nG, that is G is weakly quasi-o-minimal.

Unary functions: local monotonicity

Definition (D. Macpherson, D. Marker, C. Steinhorn)

If M is a totally ordered structures, $f(x)$ is a M-definable function, $I \subset \mathrm{dom}(f)$, then we say that f is tidy on I, if one of the following holds:

1. $\forall x \in I$ there is an infinite interval $J \subset I$ such, that $x \in J$ and f is strictly increasing on J (we say, that f is locally increasing on I).
2. $\forall x \in I$ there is an infinite interval $J \subset I$ such, that $x \in J$ and f is strictly decreasing on J (we say, that f is locally decreasing on I).
3. $\forall x \in I$ there is an infinite interval $J \subset I$ such, that $x \in J$ and f is constant on J (we say, that f is locally constant on I).

and, if for some $x \in I$ the set $\{y \in I \mid f \text{ is strictly monotonic on } (x, y) \cup (y, x)\}$ has maximum or minimum, then f is strictly monotonic on I.

Theorem (D. Macpherson, D. Marker, C. Steinhorn, 1994)

Any definable unary function in a weakly o-minimal theory is tidy.

Definable unary functions in dp-minimal ordered groups

Theorem (J. Goodrick, V. Verbovskiy, 2019)

Let G be an ordered groups whose elementary theory is dp-minimal, and let G has boundedly many definable convex subgroups. Then for any definable unary continuous function f there is a finite partition $X_1, \ldots, X_n$ of its domain, such that the restriction of f to any of X_i is locally monotone.

In
J. Goodrick. A monotonicity theorem for dp-minimal densely ordered groups, Journal of Symbolic Logic, 75(1) (2010), 221–238.
John Goodrick proved that for any definable unary function f there is a finite partition $Y_1, \ldots, Y_k$ of its domain, such that the restriction of f to any of Y_i is continuous in the induced topoly on Y_i.

So, we can omit in the theorem the condition that f is continuous.

n-tidy

Definition (B. Baizhanov, V. Verbovskiy)

We say that Φ is n-tidy on I, if the following holds:

- $\forall z \forall y \forall t\ [\Phi(M,z) = \Phi(M,y) \wedge z < t < y \to \Phi(M,z) = \Phi(M,t)]$
- $\Phi^{(n)}$ is tidy on I/E_{n-1}, where $\Phi^{(n)}(x,y) := \exists z[E_{n-1}(y,z) \wedge \Phi(x,z)]$
- $(\forall y \in I)\ E_n(I,y)/E_{n-1}$ has no minimum and maximum.
- $|I/E_n| \geq \omega$.

Where E_n is an equivalence relation on I such that

$$\begin{aligned} E_n(z,y) \quad &\Leftrightarrow \quad E_{n-1}(z,y) \vee \\ &\vee \quad [[z < y \wedge \neg E_{n-1}(z,y) \to \Phi^{(n)} \restriction [z,y]/E_{n-1} \text{ is strictly monotonic}] \wedge \\ &\wedge \quad [y < z \wedge \neg E_{n-1}(z,y) \to \Phi^{(n)} \restriction [y,z]/E_{n-1} \text{ is strictly monotonic}]] \end{aligned}$$

Here 0-tidy is tidy, $\Phi^{(0)} = \Phi$, $E_0(z,y) \Leftrightarrow z = y$.

More on local monotonicity

Definition

We say that Φ is strongly tidy on I if there exists $n \in N$ such that Φ is $(n-1)$-tidy on I and $\Phi^{(n)}$ is strictly monotonic on I/E_{n-1}. So we say that the depth of formula Φ on I equals n.

Theorem (V. Verbovskiy, 1997)

Let M be a model of a weakly o-minimal theory. Then any definable unary function f is strongly tidy.

Theorem (V. Verbovskiy, 2019)

Let G be an ordered groups whose elementary theory is dp-minimal, and let G has boundedly many definable convex subgroups. Then any definable unary function f is strongly tidy.

O-Morley rank

Definition

1. We say that Morley o-rank of a formula $\phi(x)$ inside a cut $\langle C, D\rangle$ is equal to or greater than 1 and write $RM_{\langle C,D\rangle}(\phi)\geq 1$ for this, if $\{\phi(x)\} \cup \langle C, D\rangle$ is consistent.
2. $RM_{\langle C,D\rangle}(\phi) \geq \alpha + 1$ if there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$, a cut $\langle C_1, D_1\rangle$ in $\mathcal{N}$ containing the cut $\langle C, D\rangle$ and there are infinitely many pairwise inconsistent formulae $\psi_i(x)$ with parameters in N such that the inequality $RM_{\langle C_1,D_1\rangle,\mathcal{N}}(\phi(x) \wedge \psi_i(x)) \geq \alpha$ holds.
3. If α is a limit ordinal, then $RM_{\langle C,D\rangle}(\phi) \geq \alpha$ if $RM_{\langle C,D\rangle}(\phi) \geq \beta$ for all $\beta < \alpha$.
4. $RM_{\langle C,D\rangle}(\phi) = \alpha$ if $RM_{\langle C,D\rangle}(\phi) \geq \alpha$ and $RM_{\langle C,D\rangle}(\phi) \not\geq \alpha + 1$.

Ordered groups of o-Morley rank 1

Theorem (V. Verbovskiy, 2018)

Let $\mathcal{M}$ be an ordered group whose elementary theory is o-stable, moreover, $RM_s(x = x) = 1$ for any cut $s = \langle C, D\rangle$ in $\mathcal{M}$.
Assume that $\mathcal{M}$ contains boundedly many definable convex subgroups.
Then the elementary theory of $\mathcal{M}$ is weakly o-minimal.

Theorem (V. Verbovskiy, 2018)

There exists an example of an ordered group, which is not weakly o-minimal, but which has Morley o-rank of the formula $x = x$ to be equal 1, and Morley degree at most 4. Obviously, this group has unboundedly many definable convex subgroups.

V. Verbovskiy, On ordered groups of Morley o-rank 1, Siberian Electronic Mathematical Reports, 15 (2018) 314–320.

Some problems

1. Does the monotonicity theorem for a densely ordered group, whose elementary theory is dp-minimal and which has unboundedly many definable convex subgroups, fail?
2. Does the monotonicity theorem for a densely ordered group, whose elementary theory is o-stable and which has boundedly many definable convex subgroups, holds "?
3. Is an o-ω-stable ordered field real closed?
4. Is there an o-stable expansion of a real closed field which is not weakly o-minimal?
5. Is it true that any weakly o-minimal expansion of an ordered groups has a weakly o-minimal theory if and only if this expansion has boundedly many definable convex subgroups?

The end!

Thank you for attention!