Computable vs low linear orders

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Introduction

Theorem (C. Jockusch, R. Soare, 1991)

There exists a low linear order with no a computable copy.

Theorem (R. Downey, M. Moses, 1989)

Any low discrete linear order has a computable copy.

Introduction

Downey's Research Program

To describe order properties P which guarantees that if a low linear order is such that $P(L)$ then L has a computable copy.

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Downey's Research Program

To describe order properties P which guarantees that if a low linear order is such that $P(L)$ then L has a computable copy.

Extended Research Program

To describe order properties P_n which guarantees that if a low$_n$ linear order is such that $P_n(L)$ then L has a computable copy.

Introduction

Theorem (A. Frolov, 2010 and, independently, A. Montalban, 2009)

A linear order is low iff the order has a $\mathbf{0}'$-computable copy with $\mathbf{0}'$-computable successors.

Positive results

Definition

A linear order is k-quasidiscrete, if each its block is either infinite (ω, ω^*, ζ), or contains no more than k elements.

A linear order is k-discrete, if each its block is either ζ, or contains no more than k elements.

A linear order is strongly η-like, if there is k such that each its block contains no more than k elements.

Positive results

Theorem (A. Frolov, 2010)

Any low k-quasidiscrete has a computable copy via a $\mathbf{0}'''$-computable isomorphism.

If the order is k-discrete then the isomorphism is $\mathbf{0}''$-computable.

If the order is strongly η-like then the isomorphism is $\mathbf{0}'$-computable.

Positive results

Theorem (P. Alaev, A. Frolov, J. Thurber, 2009)

Any low_2 1-quasidiscrete linear order has a computable copy via a $\mathbf{0}'''$-computable isomorphism.

Positive results

Definition

A linear order is called η-like, if it has the form $\sum_{q \in \mathbb{Q}} f(q)$, where $f : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$.

If the range of f is bounded then the order is strongly η-like (without the greatest and the least elements).

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If the range of f is bounded then the order is strongly η-like (without the greatest and the least elements).

Theorem (A. Frolov, 2006)

Any low strongly η-like has a computable copy.

Positive results

Theorem (A. Frolov, M. Zubkov, 2009)

If L is η-like then the following are equivalent:

1) L has a $\mathbf{0}'$-computable copy such that the successor and the block relations are both $\mathbf{0}'$-computable.

2) $L \cong \sum_{q \in \mathbb{Q}} f(q)$, where $f : \mathbb{Q} \to \mathbb{N} \setminus \{0\}$ and
$epigr(f) = \{(q, n) \in \mathbb{Q} \times \mathbb{N} \mid f(q) \leq n\} \in \Pi^0_2$.
It means that f is $\mathbf{0}'$-limitwise monotonic function.

3) L has a computable copy with Π^0_1 blocks.

Positive results

Theorem (A. Frolov, 2012)

If a low η-like linear order does not contain an infinite strongly η-like interval, then it has a computable copy via $\mathbf{0}''$-computable isomorphism.

Theorem (A. Frolov, 2012)

If a linear order does not contain an infinite strongly η-like interval and its condensation is η, then it has a computable copy via $\mathbf{0}''$-computable isomorphism.

Positive results

Definition

Let L is η-like linear order. The block $[x]_L$ is called a left (similarly right) local maximum, if there is $y <_L x$ and $y \notin [x]_L$ such that, for any z with $[y]_L < [z]_L < [x]_L$, $|[z]_L| < |[x]_L|$.

Theorem (M. Zubkov, 2017)

Let a linear order L be η-like such that the sizes of all left and right local maximums are bounded. If L is low then L has a computable copy.

Positive results

Definition

A final segment J is called a descending cut if J is not empty and has no least element.

Theorem (A. Kach, A. Montalban, 2011)

If a low_n linear order has only finitely many descending cuts then it has a computable copy.

Negative results

Theorem (C. Jockusch, R. Soare, 1991)

There exists a low linear order with no a computable copy.

The order has the form $\eta + 2 + \eta + L_0 + \eta + 3 + \eta + L_1 + \cdots$, where each L_i is equal to either $\omega + \omega^*$, or $\omega + k$, or $k + \omega^*$.

Negative results

Theorem (A. Frolov, 2014)

There exists a low_2 scattered linear order with no a computable copy.
Only the third condensation of the order is 1.

Definition

A linear order is called scattered, if it does not contain dense suborder.

Negative results

Theorem (A. Frolov, 2018)

There exists a low η-like linear order with no a computable copy.

Definition

A linear order is η-like, if it does not contain an infinite block.

Negative results

Theorem (A. Frolov, M. Zubkov, in preparation)

There exists a low strongly η-representation of some set with no a computable copy.

Definition

Let $A = \{a_0 < a_1 < a_2 < \cdots\}$. Then a linear order is called strongly η-representation of A, if it has the form $\eta + a_0 + \eta + a_1 + \eta + a_2 + \cdots$.

Open questions

Open question

Has a low scattered linear order a computable copy?

Open questions

Open question

Has a low scattered linear order a computable copy?

Other questions

I have no idea.

Effective categoricity

Theorem (A. Frolov, in preparation)

Let A be a 2-c.e. in and over $\mathbf{0}''$. Then there exists a computable linear order whose degree of categoricity is $\deg_T(A)$.

Definition

The degree $\mathbf{x}$ is the degree of categoricity of a structure $\mathcal{A}$ (if it exists), if $\mathcal{A}$ is $\mathbf{x}$-categorical and $\mathbf{y} \geq \mathbf{x}$ for all $\mathbf{y}$ such that $\mathcal{A}$ is $\mathbf{y}$-categorical.

Effective categoricity

Theorem (A. Frolov, in preparation)

Let A be a c.e. in and over $\mathbf{0}''$. Then there exists a computable rigid linear order whose degree of categoricity is $\deg_T(A)$.

Sketch of proof (1)

Let $Odd = \sum_{q \in \mathbb{Q}} f(q)$, where

1) $f(q_1) \neq f(q_2)$ for $q_1 \neq q_2$,
2) $rng(f) = \{1, 3, 5, 7, \ldots\}$.

Effective categoricity

Sketch of proof (2)

1) $\lambda(A) = 4 + \lambda_0 + 6 + \lambda_1 + 8 + \lambda_2 + 10 + \cdots$, where

$$\lambda_i = \begin{cases} Odd + 2 + Odd & \text{if } i \notin A \\ Odd + 2 + Odd + 2 + Odd & \text{if } i \in A \end{cases}.$$

2) A is $\Sigma^0_2(\emptyset')$. So, we can build a low copy of $\lambda(A)$.

3) There is a uniform sequence of low orders (and hence computably presentable) of order types λ_i.

Effective categoricity

Theorem (A. Frolov, in preparation)

Let A be a 2-c.e. in and over $\mathbf{0}''$. Then there exists a computable linear order whose degree of categoricity is $\deg_T(A)$.

Sketch of proof (1)

$\tau^1(X) = 4 + \tau_0 + 4 + \tau_1 + 4 + \tau_2 + 4 + \cdots$, where

$$\tau_i = \begin{cases} 3\eta + 2 + 3\eta & \text{if } i \notin X \\ 3\eta + 2 + 3\eta + 2 + 3\eta & \text{if } i \in X \end{cases}$$

X is $\Sigma^0_2(\emptyset')$.

Effective categoricity

Sketch of proof (2)

1) Let $A = X_1 - X_2$, where $X_1 \subseteq X_2$

2) $\tau^2(X_1, X_2) =$
$\tau^1(\emptyset'') + 6 + \tau_0 + 5 + \mu_0 + 6 + \tau_1 + 5 + \mu_1 + 6 + \tau_2 + 5 + \mu_2 + 6 + \cdots,$
where

$$\tau_i = \begin{cases} 3\eta + 2 + 3\eta & \text{if } i \notin X_1 \\ 3\eta + 2 + 3\eta + 2 + 3\eta & \text{if } i \in X_1 \,\&\, i \notin X_2 \\ ShS(2,3) & \text{if } i \in X_1 \,\&\, i \in X_2 \end{cases}$$

$$\mu_i = \begin{cases} 4\eta & \text{if } i \notin X_1 \\ 4\eta + 3\eta + 2 + 3\eta & \text{if } i \in X_1 \,\&\, i \notin X_2 \\ 4\eta + 3\eta + 2 + 3\eta + 2 + 3\eta & \text{if } i \in X_1 \,\&\, i \in X_2 \end{cases}$$

Effective categoricity

Theorem (Goncharov, 1975, Goncharov, Dzgoev, 1980, Remmel, 1981)

A linear order is computably categorical iff it is relative computably categorical iff it has only finitely many pairs of successors.

Definition

A computable structure $\mathcal{A}$ is called computably categorical, if any computable copy of $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$.
A computable structure $\mathcal{A}$ is called relative computably categorical, if any X-computable copy of $\mathcal{A}$ is X-computably isomorphic to $\mathcal{A}$.

Effective categoricity

Theorem (A. Frolov, in preparation)

There exists a computable $\mathbf{0}''$-categorical linear order such that it is not relatively $\mathbf{0}''$-categorical.

Sketch proof(1)

Let $A_1 \subseteq A_2$ be c.e. sets.

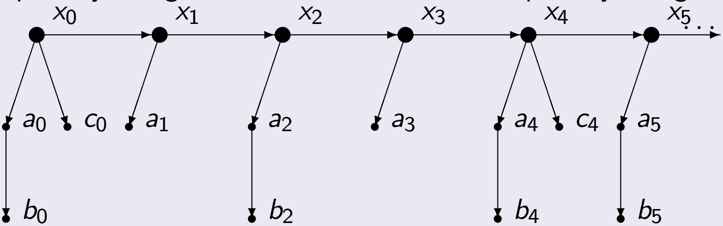
$\tau^3(A_1, A_2) = \tau^1(\emptyset'') + 5 + \tau_0 + 5 + \tau_1 + 5 + \tau_2 + 5 + \cdots$, where

$$\tau_i = \begin{cases} 4\eta & \text{if } i \notin A_1 \\ 4\eta + 3\eta + 2 + 3\eta & \text{if } i \in A_1 \,\&\, i \notin A_2 \\ 4\eta + 3\eta + 2 + 3\eta + 2 + 3\eta & \text{if } i \in A_1 \,\&\, i \in A_2 \end{cases}$$

Effective categoricity

Sketch proof(2)

There are c.e. sets $X_1 \subseteq X_2$ such that the graph $G(X_1, X_2)$ is computably categorical but is not relative computably categorical.



$0 \in X_1; 1 \notin X_2; 2 \in X_2 - X_1; 3 \notin X_2; 4 \in X_1; 5 \in X_2 - X_1$

Open questions

Open question

Does the class of all $\mathbf{0}'$-categorical linear orders coincide with the class of all relative $\mathbf{0}'$-categorical linear orders?

Open questions

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Does the class of all $\mathbf{0}'$-categorical linear orders coincide with the class of all relative $\mathbf{0}'$-categorical linear orders?

Theorem (C.F.D. McCoy, 2003)

A linear order L is relative I'-categorical iff L is a finite sum of finite orders, ω, ω^*, and $n\eta$, where L contains the limit points of $n\eta$ except for the greatest and the least elements of L.

Thanks

Thank you for your attention!