On supercompactness of $\omega_1$.

Daisuke Ikegami
(Shibaura Institute of Technology)

Joint work with Nam Trang
Set Theory Workshop in Kyoto!

November 18–22 in 2019
More picture of red leaves in Kyoto
This talk is about Set Theory.
In this talk...

We focus on the richness of $\omega_1$, the least uncountable cardinal.
The Continuum Hypothesis (CH)

Theorem (Cantor)

The real line $\mathbb{R}$ is uncountable.

Definition (Cantor)

The Continuum Hypothesis (CH) states the following:

$$|\mathbb{R}| = \omega_1.$$
The Continuum Hypothesis (CH) ctd.

**Theorem (Gödel)**

One cannot refute CH in ZFC.

Method: **Gödel’s Constructible Universe** $\mathcal{L}$

If $V \models ZF$, then $\mathcal{L} \models ZFC + CH$. 

![Diagram showing the relationship between $\text{Ord}$, $\mathcal{L}$, and $V$.](diagram.png)
The Continuum Hypothesis (CH) ctd..

Theorem (Cohen)

One cannot prove CH in ZFC.

Method: Forcing

If $V \models \text{ZFC}$, then for some $G$, $V[G] \models \text{ZFC} + \neg \text{CH}$. 
Before Cohen introduced forcing, Gödel anticipated that one cannot decide the truth-value of CH in ZFC.

Gödel’s Program

Solve “mathematically interesting” problems in a “well-justified” axiom system extending ZFC.

A candidate of such an axiom system: ZFC + large cardinal axioms
In ZFC, all the large cardinals are much bigger than $\omega_1$.

Our interest: Large cardinal properties of $\omega_1$ in ZF
Theorem

1. (Jech/Takeuti) In ZFC, if there is a measurable cardinal, then one can find a model $M$ of ZF such that $\omega_1$ is a measurable cardinal in $M$.

2. (Takeuti) One can replace “a measurable cardinal” in the above item with “a supercompact cardinal”.
Let $\kappa$ be a cardinal with $\kappa = |V_\kappa|$ in ZFC.
The model $V(\mathbb{R}^*)$ (Solovay model)

Let $\kappa$ be a cardinal with $\kappa = |V_\kappa|$ in ZFC.

Let $G$ be a $\text{Coll}(\omega, < \kappa)$-generic filter over $V$, and set $\mathbb{R}^* = \bigcup_{\alpha < \kappa} \mathbb{R}^{V[G \upharpoonright \alpha]}$. 
The model $V(R^*)$ (Solovay model)

Let $\kappa$ be a cardinal with $\kappa = |V_\kappa|$ in ZFC.

Let $G$ be a $\text{Coll}(\omega, < \kappa)$-generic filter over $V$, and set $R^* = \bigcup_{\alpha < \kappa} R^{V[G|\alpha]}$.

Let $V(R^*)$ be the least transitive proper class model $N$ of ZF such that $V \subseteq N$ and $R^* \in N$. Then
The model $V(R^*)$ (Solovay model)

Let $\kappa$ be a cardinal with $\kappa = |V_\kappa|$ in ZFC.

Let $G$ be a $\text{Coll}(\omega, < \kappa)$-generic filter over $V$, and set $R^* = \bigcup_{\alpha < \kappa} R^V[G|\alpha]$.

Let $V(R^*)$ be the least transitive proper class model $N$ of ZF such that $V \subseteq N$ and $R^* \in N$. Then

1. $\kappa = \omega_1^{V(R^*)}$ and $R^* = R^V(R^*)$. 
The model $V(R^*)$ (Solovay model)

Let $\kappa$ be a cardinal with $\kappa = |V_\kappa|$ in ZFC.

Let $G$ be a $\text{Coll}(\omega, < \kappa)$-generic filter over $V$, and set $R^* = \bigcup_{\alpha < \kappa} R^{V[G \upharpoonright \alpha]}$.

Let $V(R^*)$ be the least transitive proper class model $N$ of ZF such that $V \subseteq N$ and $R^* \in N$. Then

1. $\kappa = \omega_1^{V(R^*)}$ and $R^* = R^{V(R^*)}$,

2. in $V(R^*)$, every set of reals is Lebesgue measurable and the Lebesgue measure is $\sigma$-additive,
Let $\kappa$ be a cardinal with $\kappa = |V_\kappa|$ in ZFC.

Let $G$ be a $\text{Coll}(\omega, < \kappa)$-generic filter over $V$, and set $R^* = \bigcup_{\alpha < \kappa} R^{V[G|\alpha]}$.

Let $V(R^*)$ be the least transitive proper class model $N$ of ZF such that $V \subseteq N$ and $R^* \in N$. Then

1. $\kappa = \omega_1^{V(R^*)}$ and $R^* = R^{V(R^*)}$,

2. in $V(R^*)$, every set of reals is Lebesgue measurable and the Lebesgue measure is $\sigma$-additive,

3. DC holds in $V(R^*)$ if and only if $\kappa$ is inaccessible in $V$, and
Let $\kappa$ be a cardinal with $\kappa = |V_\kappa|$ in ZFC.

Let $G$ be a $\text{Coll}(\omega, < \kappa)$-generic filter over $V$, and set $\mathbb{R}^* = \bigcup_{\alpha < \kappa} \mathbb{R}^{V[G|\alpha]}$.

Let $V(\mathbb{R}^*)$ be the least transitive proper class model $N$ of ZF such that $V \subseteq N$ and $\mathbb{R}^* \in N$. Then

1. $\kappa = \omega_1^{V(\mathbb{R}^*)}$ and $\mathbb{R}^* = \mathbb{R}^{V(\mathbb{R}^*)}$,

2. in $V(\mathbb{R}^*)$, every set of reals is Lebesgue measurable and the Lebesgue measure is $\sigma$-additive,

3. DC holds in $V(\mathbb{R}^*)$ if and only if $\kappa$ is inaccessible in $V$, and

4. if $\kappa$ is measurable / supercompact in $V$, then so is $\kappa = \omega_1$ in $V(\mathbb{R}^*)$. 

For a set $X$, $\mathcal{P}_{\omega_1} X$ denotes the set of all countable subsets of $X$.

1. $F$ is called a filter on $\mathcal{P}_{\omega_1} X$ if $F$ consists of subsets of $\mathcal{P}_{\omega_1} X$ and
   - $\emptyset \notin F$, $\mathcal{P}_{\omega_1} X \in F$,
   - $A \in F$, $A \subseteq B \subseteq \mathcal{P}_{\omega_1} X \Rightarrow B \in F$, and
   - $A, B \in F \Rightarrow A \cap B \in F$.

2. A filter $F$ on $\mathcal{P}_{\omega_1} X$ is called an ultrafilter if for any subset $A$ of $\mathcal{P}_{\omega_1} X$, either $A$ or $(\mathcal{P}_{\omega_1} X \setminus A)$ is in $F$.

3. A filter $F$ on $\mathcal{P}_{\omega_1} X$ is $\sigma$-complete if $F$ is closed under countable intersections.
**Notation**

Let $F$ be a filter on $\mathcal{P}_{\omega_1}X$.

For a formula $\phi$, when the set $\{\sigma \in \mathcal{P}_{\omega_1}X \mid \phi(\sigma)\}$ is in $F$, we say “For $F$-measure one many $\sigma$, $\phi(\sigma)$ holds”.

**Definition (ZF)**

1. A filter $F$ on $\mathcal{P}_{\omega_1}X$ is **fine** if for any element $x$ of $X$, for $F$-measure one many $\sigma$, $x \in \sigma$.

2. A filter $F$ on $\mathcal{P}_{\omega_1}X$ is **normal** if for any function $f : \mathcal{P}_{\omega_1}X \to \mathcal{P}_{\omega_1}X$, if for $F$-measure one many $\sigma$, $f(\sigma) \cap \sigma \neq \emptyset$, then there is some $x_0 \in X$ such that for $F$-measure one many $\sigma$, $x_0 \in f(\sigma)$.

**Remark**

An ultrafilter $F$ is normal iff $F$ is closed under diagonal intersections.
\begin{itemize}
\item \(\omega_1\) is \textit{X-strongly compact} if there is a fine \(\sigma\)-complete ultrafilter on \(\mathcal{P}_{\omega_1} X\).
\item \(\omega_1\) is \textit{X-supercompact} if there is a fine \& normal \(\sigma\)-complete ultrafilter on \(\mathcal{P}_{\omega_1} X\).
\item \(\omega_1\) is \textit{measurable} if there is a non-principal \(\sigma\)-complete ultrafilter on \(\omega_1\).
\item \(\omega_1\) is \textit{supercompact} if for any set \(X\), \(\omega_1\) is \(X\)-supercompact.
\end{itemize}
Solovay: the Axiom of Determinacy (AD) implies that $\omega_1$ is measurable.

Martin: AD implies that $\omega_1$ is $\mathbb{R}$-strongly compact.

Solovay: $\text{AD}_\mathbb{R}$ implies that $\omega_1$ is $\mathbb{R}$-supercompact

while AD does NOT imply $\omega_1$ is $\mathbb{R}$-supercompact.
Woodin; Trang and Wilson: The following are equiconsistent:

1. ZFC + “There are $\omega$-many Woodin cardinals.”,
2. ZF + AD,
3. ZF + DC + $\omega_1$ is $\mathbb{R}$-strongly compact and $\neg \square_{\omega_1}$, and
4. ZF + DC + $\omega_1$ is $\mathcal{P}(\omega_1)$-strongly compact.

Trang and Wilson: The following are equiconsistent:

1. ZF + DC + AD$_\mathbb{R}$ and
2. ZF + DC + $\omega_1$ is $\mathcal{P}(\mathcal{P}(\mathbb{R}))$-strongly compact.
Recent Results

**Theorem (Woodin)**

Assume that there are proper class many Woodin cardinals which are limits of Woodin cardinals in ZFC. Then the following hold in the Chang\(^{+}\) model (\(\mathcal{C}^{+}\)):

1. \(\omega_1\) is supercompact, and
2. AD.

**Definition**

The Chang\(^{+}\) model (\(\mathcal{C}^{+}\)) is of the form \(L(\text{Ord}^\omega)[(\mathcal{F}_\gamma \mid \gamma \in \text{Ord})]\), where \(\mathcal{F}_\gamma\) is the club filter on \(\mathcal{P}_{\omega_1}(\gamma^\omega)\).

Note: \(\mathcal{C}^{+}\) is essentially different from \(V(\mathbb{R}^*)\) where AD must fail.
Our Results

We focus on the influence of the richness of $\omega_1$ on the real line:

**Theorem (I., Trang)**

Working in ZF, assume that $\omega_1$ is supercompact. Then

1. the **Axiom of Dependent Choice (DC)** holds,
Our Results

We focus on the influence of the richness of $\omega_1$ on the real line:

Theorem (I., Trang)
Working in ZF, assume that $\omega_1$ is supercompact. Then

1. the Axiom of Dependent Choice (DC) holds,
2. there is NO injection from $\omega_1$ to the reals,
Our Results

We focus on the influence of the richness of $\omega_1$ on the real line:

**Theorem (I., Trang)**

Working in ZF, assume that $\omega_1$ is supercompact. Then

1. the Axiom of Dependent Choice (DC) holds,
2. there is NO injection from $\omega_1$ to the reals,
3. in the Chang model $L(\text{Ord}^\omega)$, every set of reals is Lebesgue measurable,
## Our Results

We focus on the influence of the richness of $\omega_1$ on the real line:

### Theorem (I., Trang)

Working in ZF, assume that $\omega_1$ is supercompact. Then

1. the **Axiom of Dependent Choice** (DC) holds,
2. there is **NO** injection from $\omega_1$ to the reals,
3. in the **Chang model** $L(\text{Ord}^{\omega})$, every set of reals is Lebesgue measurable,
4. every set has a **sharp**, moreover, $AD^{L(\mathbb{R})}$ holds in any set generic extension, and
Our Results

We focus on the influence of the richness of $\omega_1$ on the real line:

**Theorem (I., Trang)**

Working in ZF, assume that $\omega_1$ is supercompact. Then

1. the **Axiom of Dependent Choice** (DC) holds,
2. there is **NO** injection from $\omega_1$ to the reals,
3. in the **Chang model** $L(\text{Ord}^\omega)$, every set of reals is Lebesgue measurable,
4. every set has a **sharp**, moreover, $\text{AD}^{L(\mathbb{R})}$ holds in any set generic extension, and
5. every **Suslin** set of reals is the projection of a relation on the reals which is determined.
For a set $X$, $\mathcal{P}_{\omega_1}X$ denotes the set of all countable subsets of $X$.

1. $F$ is called a filter on $\mathcal{P}_{\omega_1}X$ if $F$ consists of subsets of $\mathcal{P}_{\omega_1}X$ and
   - $\emptyset \notin F$, $\mathcal{P}_{\omega_1}X \in F$,
   - $A \in F$, $A \subseteq B \subseteq \mathcal{P}_{\omega_1}X \Rightarrow B \in F$, and
   - $A, B \in F \Rightarrow A \cap B \in F$.

2. A filter $F$ on $\mathcal{P}_{\omega_1}X$ is called an ultrafilter if for any subset $A$ of $\mathcal{P}_{\omega_1}X$, either $A$ or $(\mathcal{P}_{\omega_1}X \setminus A)$ is in $F$.

3. A filter $F$ on $\mathcal{P}_{\omega_1}X$ is $\sigma$-complete if $F$ is closed under countable intersections.
Notation

Let $F$ be a filter on $\mathcal{P}_{\omega_1}X$.
For a formula $\phi$, when the set $\{\sigma \in \mathcal{P}_{\omega_1}X \mid \phi(\sigma)\}$ is in $F$, we say “For $F$-measure one many $\sigma$, $\phi(\sigma)$ holds”.

Definition (ZF)

1. A filter $F$ on $\mathcal{P}_{\omega_1}X$ is **fine** if for any element $x$ of $X$, for $F$-measure one many $\sigma$, $x \in \sigma$.

2. A filter $F$ on $\mathcal{P}_{\omega_1}X$ is **normal** if for any function $f: \mathcal{P}_{\omega_1}X \to \mathcal{P}_{\omega_1}X$, if for $F$-measure one many $\sigma$, $f(\sigma) \cap \sigma \neq \emptyset$, then there is some $x_0 \in X$ such that for $F$-measure one many $\sigma$, $x_0 \in f(\sigma)$. 
Some words on the proofs

**DC from supercompcatness of $\omega_1$:**

Want: Given a nonempty set $X$ and a relation $R$ on $X$ with

$$(\forall x \in X) \ (\exists y \in X) \ (x, y) \in R,$$

one can find a function $f: \omega \to X$ such that

$$(\forall n \in \omega) \ (f(n), f(n + 1)) \in R.$$
Some words on the proofs

DC from supercompactness of $\omega_1$:

Want: Given a nonempty set $X$ and a relation $R$ on $X$ with

$$(\forall x \in X) \ (\exists y \in X) \ (x, y) \in R,$$

one can find a function $f : \omega \to X$ such that

$$(\forall n \in \omega) \ (f(n), f(n + 1)) \in R.$$

Enough to get the following:

(*) There is a countable set $\sigma \subseteq X$ such that

$$(\forall x \in \sigma) \ (\exists y \in \sigma) \ (x, y) \in R.$$
Some words on the proofs

**DC from supercompcatness of $\omega_1$:**

Want: Given a nonempty set $X$ and a relation $R$ on $X$ with

$$(\forall x \in X) \ (\exists y \in X) \ (x, y) \in R,$$

one can find a function $f: \omega \rightarrow X$ such that

$$(\forall n \in \omega) \ (f(n), f(n + 1)) \in R.$$  

Enough to get the following:

(*) There is a countable set $\sigma \subseteq X$ such that

$$(\forall x \in \sigma) \ (\exists y \in \sigma) \ (x, y) \in R.$$  

How to obtain (*): Fix a fine & normal ultrafilter $F$ on $\mathcal{P}_{\omega_1}X$ and prove that $F$-measure one many $\sigma$ witness (*).
Some words on the proofs ctd.

**DC from supercompcatness of $\omega_1$ ctd.:**

To show: For *F*-measure one many $\sigma$, $(\forall x \in \sigma) \ (\exists y \in \sigma) \ (x, y) \in R.$
DC from supercompcatness of $\omega_1$ ctd.:

To show: For $F$-measure one many $\sigma$, $(\forall x \in \sigma) \, (\exists y \in \sigma) \, (x, y) \in R$.

Sketch: Suppose NOT. Then since $F$ is an ultrafilter,

for $F$-measure one many $\sigma$, $(\exists x \in \sigma) \, (\forall y \in \sigma) \, (x, y) \notin R$. 
DC from supercompcatness of $\omega_1$ ctd.:

To show: For $F$-measure one many $\sigma$, $(\forall x \in \sigma) \ (\exists y \in \sigma) \ (x, y) \in R$.

Sketch: Suppose NOT. Then since $F$ is an ultrafilter,
for $F$-measure one many $\sigma$, $(\exists x \in \sigma) \ (\forall y \in \sigma) \ (x, y) \notin R$.

By normality of $F$, there is an $x_0$ s.t.
$(\exists \sigma)$ for $F$-measure one many $\sigma$, $(\forall y \in \sigma) \ (x_0, y) \notin R$. 

Some words on the proofs ctd.
DC from supercompacateness of $\omega_1$: 

To show: For $F$-measure one many $\sigma$, $(\forall x \in \sigma) (\exists y \in \sigma) (x, y) \in R$. 

Sketch: Suppose NOT. Then since $F$ is an ultrafilter, 
for $F$-measure one many $\sigma$, $(\exists x \in \sigma) (\forall y \in \sigma) (x, y) \notin R$. 

By normality of $F$, there is an $x_0$ s.t. 
$(\mu)$ for $F$-measure one many $\sigma$, $(\forall y \in \sigma) (x_0, y) \notin R$. 

By the assumption $(\forall x \in X) (\exists y \in X) (x, y) \in R$, there is a $y_0$ in $X$ s.t. 
$(\nu) (x_0, y_0) \in R$. 

}\end{proof}
Some words on the proofs ctd.

**DC from supercompcatness of $\omega_1$ ctd.:**

To show: For $F$-measure one many $\sigma$, $(\forall x \in \sigma) \ (\exists y \in \sigma) \ (x, y) \in R$.

**Sketch:** Suppose NOT. Then since $F$ is an ultrafilter,

for $F$-measure one many $\sigma$, $(\exists x \in \sigma) \ (\forall y \in \sigma) \ (x, y) \notin R$.

By normality of $F$, there is an $x_0$ s.t.

$(\exists x)$ for $F$-measure one many $\sigma$, $(\forall y \in \sigma) \ (x_0, y) \notin R$.

By the assumption $(\forall x \in X) \ (\exists y \in X) \ (x, y) \in R$, there is a $y_0$ in $X$ s.t.

$(\exists x)$ $(x_0, y_0) \in R$.

By fineness of $F$ and that $F$ is a filter,

$(\forall x \in \sigma), \ x_0, y_0 \in \sigma$. 
Some words on the proofs ctd.

**DC from supercompcatness of $\omega_1$ ctd.:**

To show: For $F$-measure one many $\sigma$, $(\forall x \in \sigma) \ (\exists y \in \sigma) \ (x, y) \in R$.

Sketch: Suppose NOT. Then since $F$ is an ultrafilter,

for $F$-measure one many $\sigma$, $(\exists x \in \sigma) \ (\forall y \in \sigma) \ (x, y) \notin R$.

By *normality* of $F$, there is an $x_0$ s.t.

(む) for $F$-measure one many $\sigma$, $(\forall y \in \sigma) \ (x_0, y) \notin R$.

By the assumption $(\forall x \in X) \ (\exists y \in X) \ (x, y) \in R$, there is a $y_0$ in $X$ s.t.

(じゅ) $(x_0, y_0) \in R$.

By *fineness* of $F$ and that $F$ is a *filter*,

(なん) for $F$-measure one many $\sigma$, $x_0, y_0 \in \sigma$.

But (む), (じゅ), and (なん) contradict that $F$ is a filter.
Sharps for sets from supercompactness of $\omega_1$

Step 1: Every real has a sharp.
Sharps for sets from supercompactness of $\omega_1$

Step 1: Every real has a sharp.

Point: Let $r$ be a real.
   Given a non-principal $\sigma$-complete ultrafilter $\mu$ over $\omega_1$, one can take the ultrapower of $(L[r], \in)$ via $\mu$
   and obtain a non-trivial elementary embedding $j: L[r] \rightarrow L[r]$. 
Sharps for sets from supercompactness of $\omega_1$ ctd.

Step 2: Every set has a sharp.
Sharps for sets from supercompactness of $\omega_1$ ctd.

Step 2: Every set has a sharp.

Sketch:
Let $X$ be a transitive set and $\mu$ be a fine & normal ultrafilter over $\mathcal{P}_{\omega_1}X$. 
Sharps for sets from supercompactness of $\omega_1$ ctd.

Step 2: Every set has a sharp.

Sketch:
Let $X$ be a transitive set and $\mu$ be a fine & normal ultrafilter over $\mathcal{P}_{\omega_1}X$. For each $\sigma$ in $\mathcal{P}_{\omega_1}X$, let $a_\sigma$ be the transitive collapse of $\langle \sigma, \in \rangle$ and $M_\sigma = \text{HOD}_{\sigma \cup \{\sigma\}}$. 
Sharps for sets from supercompactness of $\omega_1$ ctd.

Step 2: Every set has a sharp.

Sketch:
Let $X$ be a transitive set and $\mu$ be a fine & normal ultrafilter over $\mathcal{P}_{\omega_1} X$. For each $\sigma$ in $\mathcal{P}_{\omega_1} X$, let $a_{\sigma}$ be the transitive collapse of $(\sigma, \in)$ and $M_{\sigma} = \text{HOD}_{\sigma \cup \{\sigma\}}$.

By Step 1, $a_{\sigma}^\#$ exists and is in $M_{\sigma}$. 
Sharps for sets from supercompactness of $\omega_1$ ctd.

Step 2: Every set has a sharp.

Sketch:
Let $X$ be a transitive set and $\mu$ be a fine & normal ultrafilter over $\mathcal{P}_{\omega_1}X$. For each $\sigma$ in $\mathcal{P}_{\omega_1}X$, let $a_{\sigma}$ be the transitive collapse of $(\sigma, \in)$ and $M_{\sigma} = \text{HOD}_{\sigma \cup \{\sigma\}}$.

By Step 1, $a_{\sigma}^#$ exists and is in $M_{\sigma}$.

Consider the ultraproduct $\prod_{\sigma \in \mathcal{P}_{\omega_1}X} (M_{\sigma}, \in)/\mu$ and let $N$ be its transitive collapse.
Sharps for sets from supercompactness of $\omega_1$ ctd.

Step 2: Every set has a sharp.

Sketch:
Let $X$ be a transitive set and $\mu$ be a fine & normal ultrafilter over $\mathcal{P}_{\omega_1}X$. For each $\sigma$ in $\mathcal{P}_{\omega_1}X$, let $a_{\sigma}$ be the transitive collapse of $(\sigma, \in)$ and $M_{\sigma} = \text{HOD}_{\sigma \cup \{\sigma\}}$.

By Step 1, $a_{\sigma}^#$ exists and is in $M_{\sigma}$.

Consider the ultraproduct $\prod_{\sigma \in \mathcal{P}_{\omega_1}X} (M_{\sigma}, \in) / \mu$ and let $N$ be its transitive collapse. For each $x \in X$, let $j(x)$ be represented by $[\sigma \mapsto x]$ via $\mu$. Then $j^{-1}X$ is represented by $[\sigma \mapsto \sigma]$. 
Sharps for sets from supercompactness of $\omega_1$ ctd.

Step 2: Every set has a sharp.

Sketch:
Let $X$ be a transitive set and $\mu$ be a fine & normal ultrafilter over $\mathcal{P}_{\omega_1}X$. For each $\sigma$ in $\mathcal{P}_{\omega_1}X$, let $a_\sigma$ be the transitive collapse of $(\sigma, \in)$ and $M_\sigma = \text{HOD}_{\sigma \cup \{\sigma\}}$.

By Step 1, $a_\sigma^\#$ exists and is in $M_\sigma$.

Consider the ultraproduct $\prod_{\sigma \in \mathcal{P}_{\omega_1}X} (M_\sigma, \in)/\mu$ and let $N$ be its transitive collapse. For each $x \in X$, let $j(x)$ be represented by $[\sigma \mapsto x]$ via $\mu$. Then $j''X$ is represented by $[\sigma \mapsto \sigma]$.

By Łos’ Theorem for $\prod_{\sigma \in \mathcal{P}_{\omega_1}X} (M_\sigma, \in)/\mu$, the transitive collapse of $(j''X, \in)$, that is $X$, has a sharp in $N$. 

...
Sharps for sets from supercompactness of $\omega_1$ ctd.

Step 2: Every set has a sharp.

Sketch:
Let $X$ be a transitive set and $\mu$ be a fine & normal ultrafilter over $\mathcal{P}_{\omega_1}X$. For each $\sigma$ in $\mathcal{P}_{\omega_1}X$, let $a_\sigma$ be the transitive collapse of $(\sigma, \in)$ and $M_\sigma = \text{HOD}_{\sigma \cup \{\sigma\}}$.

By Step 1, $a^\#_\sigma$ exists and is in $M_\sigma$.

Consider the ultraproduct $\prod_{\sigma \in \mathcal{P}_{\omega_1}X} (M_\sigma, \in)/\mu$ and let $N$ be its transitive collapse. For each $x \in X$, let $j(x)$ be represented by $[\sigma \mapsto x]$ via $\mu$. Then $j``X$ is represented by $[\sigma \mapsto \sigma]$.

By Łos' Theorem for $\prod_{\sigma \in \mathcal{P}_{\omega_1}X} (M_\sigma, \in)/\mu$, the transitive collapse of $(j``X, \in)$, that is $X$, has a sharp in $N$. Therefore $X^\#$ exists in $N$ and so does in $V$. 
Open Questions

1. Assume that $\omega_1$ is supercompact in ZF. Then is every Suslin set of reals determined?

2. What is the consistency strength of ZF+ "$\omega_1$ is supercompact"?
THE END.