On supercompactness of ω_1.

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Joint work with Nam Trang

Set Theory Workshop in Kyoto!

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More picture of red leaves in Kyoto

This talk is about Set Theory.

In this talk...

We focus on the richness of ω_1,
the least uncountable cardinal.

The Continuum Hypothesis (CH)

Theorem (Cantor)

The real line $\mathbb{R}$ is uncountable.

Definition (Cantor)

The Continuum Hypothesis (CH) states the following:

$$|\mathbb{R}| = \omega_1.$$

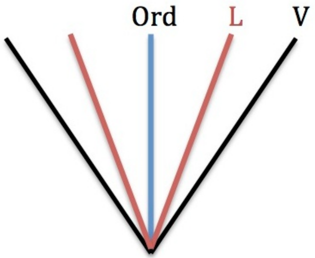
The Continuum Hypothesis (CH) ctd.

Theorem (Gödel)

One cannot refute CH in ZFC.

Method: Gödel's Constructible Universe L

If $\mathrm{V} \models$ ZF, then $\mathrm{L} \models$ ZFC+CH.



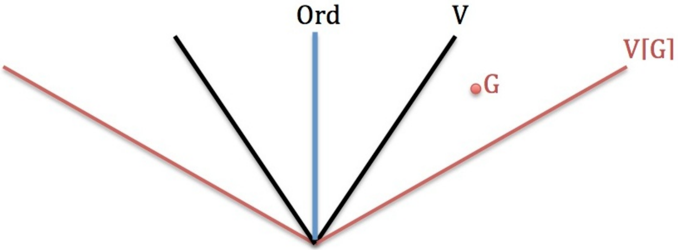
The Continuum Hypothesis (CH) ctd..

Theorem (Cohen)

One cannot prove CH in ZFC.

Method: Forcing

If $\mathrm{V} \models \mathrm{ZFC}$, then for some G, $\mathrm{V}[G] \models \mathrm{ZFC}+\neg\mathrm{CH}$.



Gödel's Program and Large Cardinals

Before Cohen introduced forcing, Gödel anticipated that one canNOT decide the truth-value of CH in ZFC.

Gödel's Program

Solve "mathematically interesting" problems in a "well-justified" axiom system extending ZFC.

A candidate of such an axiom system: ZFC + large cardinal axioms

The topic in this talk

In ZFC, all the large cardinals are much bigger than ω_1.

Our interest: Large cardinal properties of ω_1 in ZF

Background 1

Theorem

1. (Jech/Takeuti) In ZFC, if there is a measurable cardinal, then one can find a model M of ZF such that ω_1 is a measurable cardinal in M.
2. (Takeuti) One can replace "a measurable cardinal" in the above item with "a supercompact cardinal".

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3. DC holds in $\mathrm{V}(\mathbb{R}^*)$ if and only if κ is inaccessible in V, and
4. if κ is measurable / supercompact in V, then so is $\kappa = \omega_1$ in $\mathrm{V}(\mathbb{R}^*)$.

Definitions & Notations

Definition (ZF)

For a set X, $\mathcal{P}_{\omega_1}X$ denotes the set of all countable subsets of X.

1. F is called a filter on $\mathcal{P}_{\omega_1}X$ if F consists of subsets of $\mathcal{P}_{\omega_1}X$ and
   - $\emptyset \notin F$, $\mathcal{P}_{\omega_1}X \in F$,
   - $A \in F$, $A \subseteq B \subseteq \mathcal{P}_{\omega_1}X \Rightarrow B \in F$, and
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2. A filter F on $\mathcal{P}_{\omega_1}X$ is called an ultrafilter if for any subset A of $\mathcal{P}_{\omega_1}X$, either A or $\left(\mathcal{P}_{\omega_1}X \setminus A\right)$ is in F.
3. A filter F on $\mathcal{P}_{\omega_1}X$ is σ-complete if F is closed under countable intersections.

Definitions & Notations ctd.

Notation

Let F be a filter on $\mathcal{P}_{\omega_1}X$.
For a formula ϕ, when the set $\{\sigma \in \mathcal{P}_{\omega_1}X \mid \phi(\sigma)\}$ is in F, we say "For F-measure one many σ, $\phi(\sigma)$ holds".

Definition (ZF)

1. A filter F on $\mathcal{P}_{\omega_1}X$ is fine if for any element x of X, for F-measure one many σ, $x \in \sigma$.
2. A filter F on $\mathcal{P}_{\omega_1}X$ is normal if for any function $f \colon \mathcal{P}_{\omega_1}X \to \mathcal{P}_{\omega_1}X$, if for F-measure one many σ, $f(\sigma) \cap \sigma \neq \emptyset$, then there is some $x_0 \in X$ such that for F-measure one many σ, $x_0 \in f(\sigma)$.

Remark

An ultrafilter F is normal iff F is closed under diagonal intersections.

Definitions & Notations ctd..

Definition (ZF)

1. ω_1 is *X-strongly compact* if there is a fine σ-complete ultrafilter on $\mathcal{P}_{\omega_1}X$.

2. ω_1 is *X-supercompact* if there is a fine & *normal* σ-complete ultrafilter on $\mathcal{P}_{\omega_1}X$.

3. ω_1 is *measurable* if there is a non-principal σ-complete ultrafilter on ω_1.

4. ω_1 is *supercompact* if for any set X, ω_1 is X-supercompact.

Background 2

Solovay: the Axiom of Determinacy (AD) implies that ω_1 is measurable.

Martin: AD implies that ω_1 is $\mathbb{R}$-strongly compact.

Solovay: $\mathrm{AD}_{\mathbb{R}}$ implies that ω_1 is $\mathbb{R}$-supercompact

while AD does NOT imply ω_1 is $\mathbb{R}$-supercompact.

Background 2 ctd.

Woodin; Trang and Wilson: The following are equiconsistent:

1. ZFC + "There are ω-many Woodin cardinals.",
2. ZF + AD,
3. ZF + DC + ω_1 is $\mathbb{R}$-strongly compact and $\neg\Box_{\omega_1}$, and
4. ZF + DC + ω_1 is $\mathcal{P}(\omega_1)$-strongly compact.

Trang and Wilson: The following are equiconsistent:

1. ZF + DC + $AD_{\mathbb{R}}$ and
2. ZF + DC + ω_1 is $\mathcal{P}(\mathcal{P}(\mathbb{R}))$-strongly compact.

Recent Results

Theorem (Woodin)

Assume that there are proper class many Woodin cardinals which are limits of Woodin cardinals in ZFC.
Then the following hold in the Chang$^+$ model ($\mathcal{C}^+$):

1. ω_1 is supercompact, and
2. AD.

Definition

The Chang$^+$ model ($\mathcal{C}^+$) is of the form $\mathrm{L}(\mathrm{Ord}^\omega)[(\mathcal{F}_\gamma \mid \gamma \in \mathrm{Ord})]$, where $\mathcal{F}_\gamma$ is the club filter on $\mathcal{P}_{\omega_1}(\gamma^\omega)$.

Note: $\mathcal{C}^+$ is essentially different from $\mathrm{V}(\mathbb{R}^*)$ where AD must fail.

Our Results

We focus on the influence of the richness of ω_1 on the real line:

Theorem (I., Trang)

Working in ZF, assume that ω_1 is supercompact. Then

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5. every Suslin set of reals is the projection of a relation on the reals which is determined.

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DC from supercompcatness of ω_1:

Want: Given a nonempty set X and a relation R on X with

$$(\forall x \in X)\ (\exists y \in X)\ (x, y) \in R,$$

one can find a function $f \colon \omega \to X$ such that

$$(\forall n \in \omega)\ \big(f(n), f(n+1)\big) \in R.$$

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How to obtain $(*)$: Fix a fine & normal ultrafilter F on $\mathcal{P}_{\omega_1} X$
and prove that F-measure one many σ witness $(*)$.

DC from supercompcatness of ω_1 ctd.:

To show: For F-measure one many σ, $(\forall x \in \sigma)\ (\exists y \in \sigma)\ (x, y) \in R$.

Some words on the proofs ctd.

DC from supercompcatness of ω_1 ctd.:

To show: For F-measure one many σ, $(\forall x \in \sigma)\ (\exists y \in \sigma)\ (x, y) \in R$.

Sketch: Suppose NOT. Then since F is an ultrafilter,
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By fineness of F and that F is a filter,
(ん) for F-measure one many σ, $x_0, y_0 \in \sigma$.

But (む), (じゅ), and (ん) contradict that F is a filter.

Sharps for sets from supercompactness of ω_1

Step 1: Every real has a sharp.

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Point: Let r be a real.

Given a non-principal σ-complete ultrafilter μ over ω_1,

one can take the ultrapower of $(\mathrm{L}[r], \in)$ via μ

and obtain a non-trivial elementary embedding $j \colon \mathrm{L}[r] \to \mathrm{L}[r]$.

Sharps for sets from supercompactness of ω_1 ctd.

Step 2: Every set has a sharp.

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Let X be a transitive set and μ be a fine & normal ultrafilter over $\mathcal{P}_{\omega_1} X$.

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Sketch:
Let X be a transitive set and μ be a fine & normal ultrafilter over $\mathcal{P}_{\omega_1}X$.
For each σ in $\mathcal{P}_{\omega_1}X$, let a_σ be the transitive collapse of $(\sigma, \in)$
and $M_\sigma = \mathrm{HOD}_{\sigma \cup \{\sigma\}}$.

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By Step 1, $a_\sigma^\#$ exisits and is in M_σ.

Consider the ultraproduct $\prod_{\sigma \in \mathcal{P}_{\omega_1}X}(M_\sigma, \in)/\mu$ and let N be its transitive collapse.

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Consider the ultraproduct $\prod_{\sigma\in\mathcal{P}_{\omega_1}X}(M_\sigma, \in)/\mu$ and let N be its transitive collapse. For each $x \in X$, let $j(x)$ be represented by $[\sigma \mapsto x]$ via μ. Then $j``X$ is represented by $[\sigma \mapsto \sigma]$.

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By Łos' Theorem for $\prod_{\sigma \in \mathcal{P}_{\omega_1} X} (M_\sigma, \in)/\mu$,
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Then $j``X$ is represented by $[\sigma \mapsto \sigma]$.

By Łos' Theorem for $\prod_{\sigma\in\mathcal{P}_{\omega_1}X}(M_\sigma, \in)/\mu$,
the transitive collapse of $(j``X, \in)$, that is X, has a sharp in N.
Therefore $X^\#$ exists in N and so does in V.

Open Questions

1. Assume that ω_1 is supercompact in ZF. Then is every Suslin set of reals determined?
2. What is the consistency strength of ZF+ "ω_1 is supercompact"?

THE END.