Automatic structures

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Goal

Rich class of infinite structures amenable to an algorithmic treatment, in particular,

- finitely presented and
- many properties uniformly decidable.

Computable structures

Definition

A graph $(V; E)$ is computable if $V \subseteq \mathbb{N}$ and $E \subseteq V \times V \subseteq \mathbb{N}^2$ are decidable, i.e., a computable graph is finitely presented by a pair of Turing machines that decide V and E, resp.

Basic problems with this class

- first-order theory undecidable: there exists a computable graph whose first-order theory is Δ^0_ω-complete.
- natural problems are highly undecidable: isomorphism problem is Σ^1_1-complete

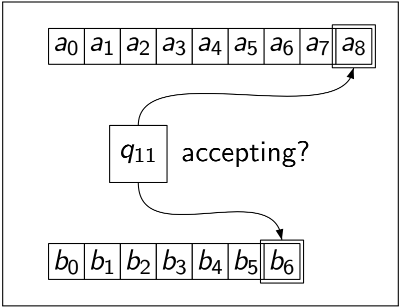
Possible solution

restrict class by, e.g., restricing class of admissible presentations – how far?

polynomial time is too powerful

For any computable graph G, there exists an isomorphic one $G' = (V'; E')$ such that V' and E' are both decidable in polynomial time (and a presentation of G' can be computed from one of G).

Multitape automata



Some properties

- accept relations on Γ^*, emptiness decidable
- effective closure under union, projection, cylindrification
- not closed under complementation, intersection; universality undecidable

Rational graphs

Definition (Morvan '00)

A graph $(V; E)$ is rational if $V \subseteq \Sigma^*$ is regular (i.e., accepted by some 1-tape automaton) and $E \subseteq V \times V \subseteq \Sigma^* \times \Sigma^*$ is accepted by some multitape automaton.

clear

rational graphs form a (proper) subclass of all computable graphs (up to isomorphism).

Example subword order

$V = \{a, b\}^*$ all words – clearly regular
$E = \{(u, v) \mid u \text{ is scattered subword of } v\}$
– accepted by 2-tape automaton with one state

Karandikar, Schnoebelen '15: Σ_2-theory of $(V; E)$ is undecidable.
$\Rightarrow$ restriction of class of computable structures to rational ones does not suffice.

Foundation

- Computable structures
- Rational graphs
- Automatic structures

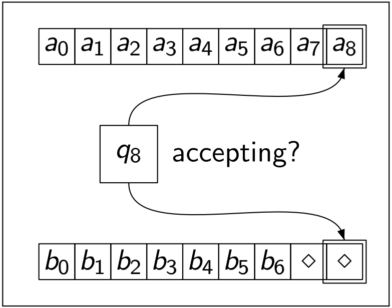
Logical Theories

- Decidability and interpretations
- Complexity

Isomorphism

Summary

Synchronous multitape automata



relation accepted by M: $R(M)$

$R \subseteq (\Gamma^*)^k$ automatic if it is accepted by some synchronous k-head automaton

Some properties of automatic relations

- emptiness and universality decidable
- effective closure under union, projection, cylindrification, complementation, intersection

Automatic structures

Definition (Hodgson '82, Khoussainov, Nerode '95)

A relational structure $(V, (R_i)_{1 \leq i \leq n})$ is

1. regular, if $V \subseteq \Gamma^*$ and $R_i \subseteq V^k \subseteq (\Gamma^*)^k$ can be accepted by synchronous k-tape automata M and M_i, resp.
   regular structure $\mathcal{A}(P)$ finitely presented by presentation $P = (M, (M_i)_{1 \leq i \leq n})$
2. automatic, if it is isomorphic to some regular structure.

Examples of automatic structures

- all finite structures
- Presburger arithmetic $(\mathbb{N}, +)$
- $(\mathbb{Q}, \leq)$ – even automatic-homogeneous (K '03), $(\mathbb{Z}, \leq)$, $(\mathbb{N}, \leq)$
- complete binary tree with equal-level predicate

Examples of automatic structures (continued)

- rewrite graph $(\Gamma^*, \to)$ of semi-Thue system
- configuration graph of Turing machines
- configuration graph with reachability $(Q\Gamma^*, \to, \to^*)$ of pushdown automata
- Cayley-graphs of "automatic" monoids, in particular of
  - rational monoids (Sakarovitch '87)
  - virtually free f.g., virtually free Abelian f.g., and hyperbolic groups (Epstein et al. '92)
  - graph products of such monoids (Fohry, K '05)
- Cayley-graphs of f.g. 2-nilpotent groups, Baumslag-Solitar groups $B(1, n)$, many metabelian groups (Kharlampovich, Khoussainov, Miasnikov '11)
- finite disjoint unions of automatic structures
  quotients of automatic structures wrt. automatic congruence (Khoussainov, Nerode '95)

Examples of non-automatic structures

- Skolem arithmetic $(\mathbb{N}, \cdot)$ (Blumensath '99)
- $(\mathbb{Z}, +)^{\aleph_0}$, Rado's graph, countable universal homogeneous partial order, countable atomless Boolean algebra (Khoussainov, Nies, Rubin, Stephan '04)
- $(\mathbb{Q}, +)$ (Tsankov '11)

- every automatic structure can be interpreted in the complete binary tree with equal-level predicate (Blumensath '99)
- automatic linear orders have finite condensation rank
  automatic order trees have finite Cantor-Bendixon-rank (Khoussainov, Rubin, Stephan '03)

Characterisations of automatic structures

- ordinal α automatic iff $\alpha < \omega^\omega$
  (Delhommé, Goranko, Knapik '03)
- $\mathcal{B}$ = Boolean algebra of (co-)finite subsets of $\mathbb{N}$
  infinite Boolean algebra automatic iff $\mathcal{B}^n$ for some $n \in \mathbb{N}$
- integral domain automatic iff finite
  (Khoussainov, Nies, Rubin, Stephan '04)
- f.g. group automatic iff virtually Abelian
  (Oliver, Thomas '05)

Theorem (Bazhenov, Harrison-Trainor, Kalimullin, Melnikov, Ng '19)
Automaticity of a given computable structure is Σ^1_1-complete.

Decidability

Theorem (Büchi '60, Hodgson '82)

There is an algorithm with:
Input: presentation P and first-order formula $\varphi(\overline{x})$
Output: synchronous multi-tape automaton M_φ with $L(M_\varphi) = \{\overline{a} \mid \mathcal{A}(P) \models \varphi(\overline{a})\}$.

space N-fold exponential in $|P| + |\varphi|$ for $\varphi \in \Sigma_{N+1}$

Proof

set of relations accepted by synchronous multi-tape automata is effectively closed under cylindrification, Boolean operations, and projections

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Theorem (Zetzsche, K, Lohrey '17)

Let $L = \{w \mid |w|_a = |w|_b\} \subseteq \{a,b\}^*$ and $N \in \mathbb{N}$.
There exists an automatic structure $\mathcal{A}_N$ such that some language $K \notin \Sigma^0_N$ is definable in $(\mathcal{A}_N, L)$.

Corollary

- The class of automatic structures is effectively closed under first-order interpretations.
- The first-order theory of automatic structures is uniformly decidable.

Remark

Holds likewise for extension of first-order logic with quantifiers

- $\exists^{\infty}$ (Blumensath '99), $\exists^{(p,q)}$ (Khoussainov, Rubin, Stephan '04), and Ramsey quantifier (Rubin '08)
- second-order quantifier "$\exists R$ infinite" provided R occurs only negatively (K, Lohrey '08)
- boundedness-quantifier (K '11)

but not for second-order logic, fixpoint logic, ...

Recall

first-order theory of automatic structures is uniformly decidable.

decision procedure by Büchi and Hodgson uses space N-fold exponential in $|P| + |\varphi|$ for automatic presentation P and first-order formula $\varphi \in \Sigma_{N+1}$

Questions

- Is this high complexity optimal?
  If so: Is it caused by structure or by formula?
- Are there classes of automatic structures with lower complexity?
- Are there classes of automatic structures where procedure by Büchi and Hodgson performs better than anticipated?

Structures of high complexity

Theorem (Blumensath '99)

There is an automatic structure whose theory is non-elementary, i.e., not in N-EXPSPACE for any $N \in \mathbb{N}$.

Theorem (K '11)

- $\exists$ automatic structure $\mathcal{A}$
  $\forall N \in \mathbb{N}$: Σ_{N+1}-theory of $\mathcal{A}$ is N-EXPSPACE-complete
- $\forall N \in \mathbb{N} \, \exists \varphi_N \in \Sigma_{N+1}$
  $\{P \text{ presentation} \mid \mathcal{A}(P) \models \varphi_N\}$ is N-EXPSPACE-complete

Structures of low and intermediate complexity

Theorem

- The theory of every automatic structure of bounded degree is in 2-EXPSPACE and this is optimal. (K, Lohrey '11)
- For automatic structures of bounded degree, the algorithm by Büchi and Hodgson runs uniformly in three-fold exponential time (ditto for $(\mathbb{N}, +)$). (Durand-Gasselin, Habermehl '12)
- The theory of every automatic structure with bounded universe (i.e., subset of $w_1^* w_2^* \cdots w_n^*$ for some words $w_1, w_2, \cdots, w_n$) is in PSPACE. (Wiemuth '15)
- For all $N \in \mathbb{N}$, there exists an automatic structure $\mathcal{A}$ whose theory is hard for N-EXPTIME and in N-EXPSPACE. (Abu Zaid, K, Lindner '18)

Decidable isomorphism problems

Theorem

Isomorphism is decidable for automatic

- ordinals (Delhommé, Goranko, Knapik '03)
- Boolean algebras and integral domains (Khoussainov, Nies, Rubin, Stephan '04)
- finitely generated groups (Oliver, Thomas '05)

Proof idea:

- we have classifications of the automatic structures in these classes,
- these classifications can be expressed in some extension of first-order logic,
- and these logics give rise to uniformly decidable theories. □

Arithmetical isomorphism problems

Theorem (Rubin '04)

Isomorphism is Π^0_3-complete for locally finite automatic structures.

Theorem

Isomorphism is Π^0_1-complete for automatic equivalence structures.

Proof idea: membership in Π^0_1 (Rubin '04):
$(A,\sim) \cong (B,\sim) \iff$
$\forall m \in \mathbb{N}, n \in \mathbb{N} \cup \{\infty\}$:
$(A,\sim)$ has $\geq m$ equivalence classes of size n iff $(B,\sim)$ does.
this property of (m,n) is expressible in FO+$\exists^\infty$
and the FO+$\exists^\infty$-theory of automatic structures is uniformly decidable.

hardness for Π^0_1 (K, Liu, Lohrey '13):
reduction from complement of Hilbert's 10th problem □

Theorem (K, Liu, Lohrey '13)

Isomorphism is Π^0_1-complete for automatic forests of height 1.

Proof idea: transform equivalence structures to forests of height 1 (and vice versa): one new root for every equivalence class □

by induction, this can be extended to

Theorem (K, Liu, Lohrey '13)

Isomorphism is Π^0_{2n-1}-complete for automatic forests of height $n \geq 1$.

Σ_1^1-complete isomorphism problems

Theorem (Khoussainov, Nies, Rubin, Stephan '07, Nies '07)
Isomorphism is Σ_1^1-complete for automatic

- successor trees
- structures
- undirected graphs
- commutative monoids
- partial orders (of height 2)
- lattices (of height 4)
- unary functions

Theorem (K, Liu, Lohrey '13)

Isomorphism is Σ^1_1-complete for automatic order trees and linear orders.

Theorem and Question

Isomorphism is in Π^0_ω for automatic scattered linear orders.

Is it decidable/arithmetical/Π^0_ω-complete?

Summary

Automatic structures form a rich class of computable structures

- no classification possible (since automaticity is Σ^1_1-complete)
- complexity of theories spans the whole range from PSPACE to non-elementary
- isomorphism spans the whole arithmetical hierarchy and contains Σ^1_1-complete instances.

Summary

Automatic structures form a rich class of computable structures amenable to an algorithmic treatment, i.e., many properties are uniformly decidable

all properties expressible in extension of first-order logic by

- $\exists^{\infty}$
- $\exists^{(p,q)}$
- Ramsey quantifier
- boundedness quantifier
- some restricted form of second-order quantification over infinite relations

Summary

Automatic structures form a rich class of computable structures amenable to an algorithmic treatment, i.e., many properties are uniformly decidable

for instance

- existence of an infinite clique
- non-wellfoundedness of an order tree
- local-finiteness of a graph
- bounded degree of a graph

Summary

Automatic structures form a rich class of computable structures amenable to an algorithmic treatment, i.e., many properties are uniformly decidable, but others are not.

- isomorphism
- reachability properties etc.
- rigidity
- existence of Eulerian or Hamiltonian path, colorability

Summary

Automatic structures form a rich class of computable structures amenable to an algorithmic treatment, i.e., many properties are uniformly decidable, but others are not. They enjoy many open questions.

- Classify the automatic members of more classes of structures. In particular: find methods to prove non-automaticity.
- Determine complexity of isomorphism problem for more classes of automatic structures (e.g., scattered linear orders).
- Find classes where the isomorphism problem is complete for levels of the hyperarithmetical hierarchy.
- Investigate automatic structures with restricted universes (unary, bounded, polynomial growth, ...).

Summary

Automatic structures form a rich class of computable structures amenable to an algorithmic treatment, i.e., many properties are uniformly decidable, but others are not. They enjoy many open questions.

Theorem (K, Liu, Lohrey '13)

Elementary equivalence of automatic (equivalence) structures is Π^0_1-complete.

Question

Identify classes of automatic structures with decidable elementary equivalence. Which complexities are possible?