CLASSIFICATION OF COUNTABLE MODELS OF COMPLETE THEORIES AND ITS APPLICATIONS

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Theories, Models, and Types

- (Elementary) Theory T: an information written by first-order formulas / **syntax**
- Complete Theory: a maximal consistent information
- Model $\mathcal{M}$ of T: an object realizing T / **semantic**
- Countable Model (Structure): a model (of a theory) with countably many elements
- (Complete) type: a (complete) information about a finite set A in $\mathcal{M}$

Spectrum functions

$I(T,\lambda)$ denotes the number of pairwise non-isomorphic models of T and having λ elements (λ is an infinite power, T is a complete theory without finite models).

The *spectrum function* $I(T,\cdot)$ maps a (finite or infinite) power $I(T,\lambda)$ for an infinite power λ.

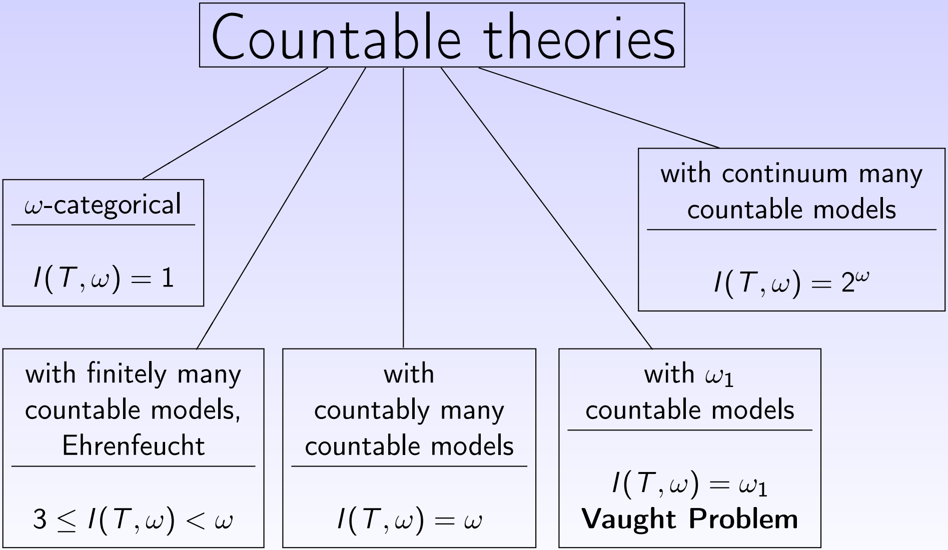
We consider $\lambda = \omega$ (countable, i.e., enumerable by natural numbers forming the set ω).

It is known that for any countable complete theory T,

- $I(T,\omega) \neq 2$, **Vaught Theorem**,
- if $I(T,\omega) > \omega_1$ then $I(T,\omega) = 2^{\omega}$, **Morley Theorem**.

Thus, $I(T,\omega) \in (\omega \setminus \{0,2\}) \cup \{\omega, \omega_1, 2^{\omega}\}$.

Countable theories with respect to countable models



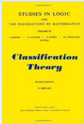
Theories

Greece, Athens, Acropolis, Theorias street

Models

Models

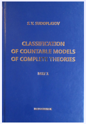
uncountable

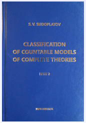
Shelah's 

,
B.Hart, E.Hrushovski,
M. S. Laskowski,
and many other specialists

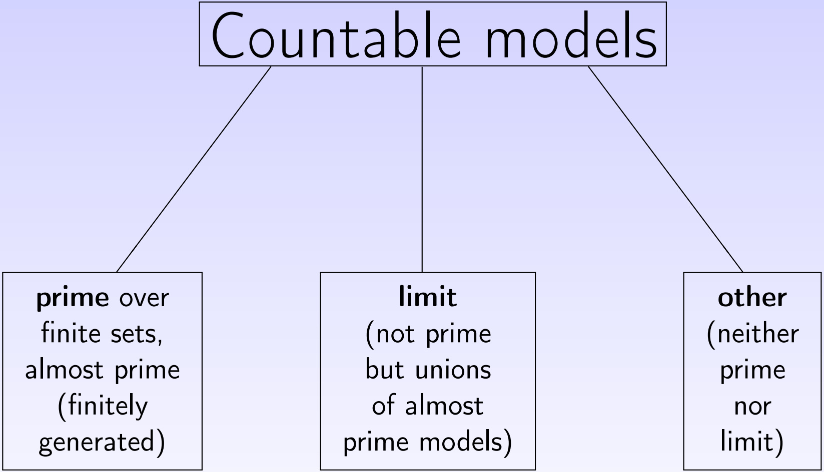
countable

Many results
by specialists
including the author's





Countable models



Countable models and types

We denote by

- $P(T)$ the number of pairwise non-isomorphic (almost) prime models of T,
- $L(T)$ the number of pairwise non-isomorphic limit models of T,
- $\mathrm{NPL}(T)$ the number of pairwise non-isomorphic other countable models of T.

The set of all types of theory T is denoted by $S(T)$.

Countable theories and types

Countable theories

small, i.e., with countably many types

$|S(T)| = \omega$

unsmall, i.e., with continuum many types

$|S(T)| = 2^\omega$

Countable models of small theories

Small theories

$\Rightarrow \mathrm{NPL}(T) = 0$

- $I(T,\omega) = 1$
  - $P(T) = 1$
  - $L(T) = 0$
- $3 \leq I(T,\omega) < \omega$
  - $1 < P(T) < \omega$
  - $1 \leq L(T) < \omega$
- $I(T,\omega) = \omega$
  - $1 < P(T) \leq \omega$
  - $1 \leq L(T) \leq \omega$
  - $P(T) + L(T) = \omega$
- $I(T,\omega) = \omega_1$
  - $1 < P(T) \leq \omega$
  - $L(T) = \omega_1$, existence of T is unknown
- $I(T,\omega) = 2^\omega$
  - $1 < P(T) \leq \omega$
  - $L(T) = 2^\omega$

Triples for distributions of countable models of theories with continuum many types[1]

THEOREM (R.A. Popkov, S.V. Sudoplatov, 2015)

Assuming the continuum hypothesis, for any theory T in the class $\mathcal{T}_c$ of theories with continuum many types, the triple $\mathrm{cm}_3(T) = (P(T), L(T), \mathrm{NPL}(T))$ has one of the following values:
(1) $(2^\omega, 2^\omega, \lambda)$, *where* $\lambda \in \omega \cup \{\omega, 2^\omega\}$;
(2) $(0, 0, 2^\omega)$;
(3) $(\lambda_1, \lambda_2, 2^\omega)$, *where* $\lambda_1 \geq 1$, $\lambda_1, \lambda_2 \in \omega \cup \{\omega, 2^\omega\}$.
All these values have realizations in the class $\mathcal{T}_c$.

[1] R.A. Popkov, S.V. Sudoplatov, Distributions of countable models of complete theories with continuum many types, Siberian Electronic Mathematical Reports. 2015. Vol. 12. P. 267–291.

Classification of countable models with respect to two basic characteristics

We consider two basic characteristics for the classification of countable models of a theory:

- Rudin–Keisler preorders $\leq_{\mathrm{RK}}$ for isomorphism types of almost prime models:

$$\mathcal{M}(A) \leq_{\mathrm{RK}} \mathcal{M}(B) \Leftrightarrow \mathcal{M}(B) \text{ realizes } \mathrm{tp}(A);$$

- distributions of limit models over equivalence classes of almost prime models.

We obtain a classification for the class of small theories, with respect to these characteristics.

The classification is generalized for the class $\mathcal{T}_c$ (with R.A. Popkov).

Notations

- $\sim_{RK} \rightleftharpoons \leq_{\mathrm{RK}} \cap \leq_{\mathrm{RK}}^{-1}$;
- $\widetilde{\mathbf{M}}$ is the $\sim_{RK}$-class containing the isomorphism type $\mathbf{M}$ for a prime model over a finite set;
- $\mathrm{IL}(\widetilde{\mathbf{M}})$ is the number of limit models being unions of elementary chains of models with isomorphism types in $\widetilde{\mathbf{M}}$.

Characterization of $I(T,\omega) < \omega$ with respect to limit models[2] [3]

Theorem

For any countable complete theory T, the following conditions are equivalent:
(1) $I(T,\omega) < \omega$;
(2) T is small, $|\mathrm{RK}(T)| < \omega$ and $\mathrm{IL}(\widetilde{\mathbf{M}}) < \omega$ for any $\widetilde{\mathbf{M}} \in \mathrm{RK}(T)/\sim_{\mathrm{RK}}$.

[2] Sudoplatov S. V. Complete theories with finitely many countable models. I // Algebra and Logic. — 2004. — Vol. 43, No. 1. — P. 62–69.

[3] Sudoplatov S. V. Classification of Countable Models of Complete Theories. — Novosibirsk : NSTU, 2014, 2018.

Characterization of $I(T,\omega) < \omega$ with respect to limit models

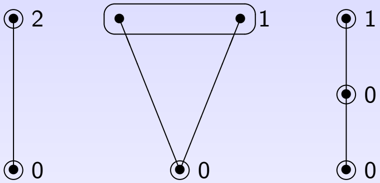
If (1) or (2) holds then T possesses the following properties:
(a) $\mathrm{RK}(T)$ has a least element $\mathbf{M}_0$ (an isomorphism type of a prime model) and $\mathrm{IL}(\widetilde{\mathbf{M}_0}) = 0$;
(b) $\mathrm{RK}(T)$ has a greatest $\sim_{\mathrm{RK}}$-class $\widetilde{\mathbf{M}_1}$ (a class of isomorphism types of all prime models over realizations of powerful types) and $|\mathrm{RK}(T)| > 1$ implies $\mathrm{IL}(\widetilde{\mathbf{M}_1}) \geq 1$;
(c) if $|\widetilde{\mathbf{M}}| > 1$ then $\mathrm{IL}(\widetilde{\mathbf{M}}) \geq 1$.
Moreover, the following *decomposition formula* holds:

$$I(T,\omega) = |\mathrm{RK}(T)| + \sum_{i=0}^{|\mathrm{RK}(T)/\sim_{\mathrm{RK}}|-1} \mathrm{IL}(\widetilde{\mathbf{M}_i}), \qquad (1)$$

where $\widetilde{\mathbf{M}_0}, \ldots, \widetilde{\mathbf{M}_{|\mathrm{RK}(T)/\sim_{\mathrm{RK}}|-1}}$ are all elements of the partially ordered set $\mathrm{RK}(T)/\sim_{\mathrm{RK}}$.

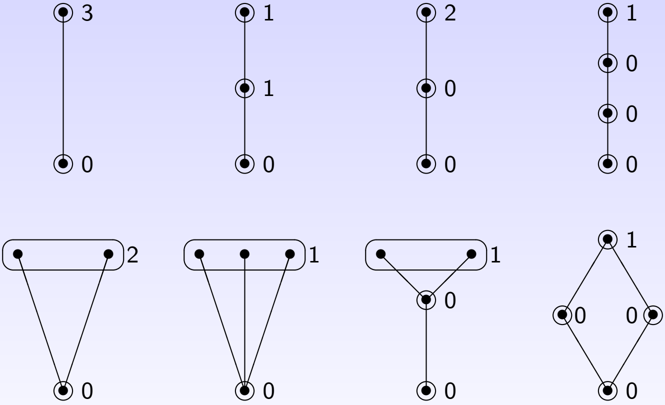
Examples of diagrams for Ehrenfeucht theories

$I(T, \omega) = 3$



$I(T, \omega) = 4$

Examples of diagrams for Ehrenfeucht theories



$$I(T,\omega)=5$$

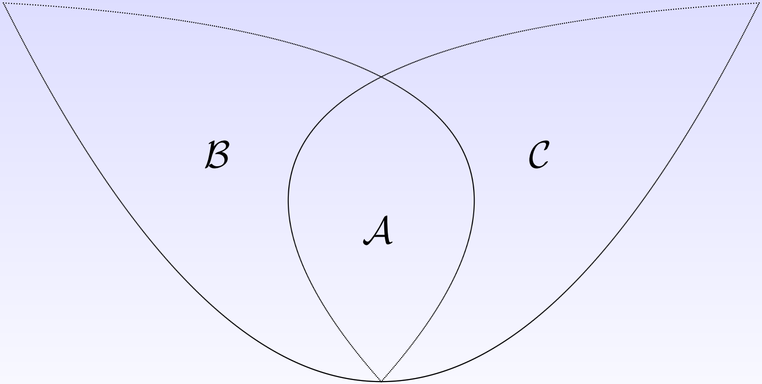
Realizing basic characteristics for the classification, we use syntactic generalizations of semantic Jonsson–Fraïssé–Hrushovski–Herwig generic constructions, based on syntactic amalgams.

Let $\Phi(A)$, $\Psi(B)$, $\mathrm{X}(C)$ be types (diagrams) in a class $\mathbf{T}_0$, describing links between elements in finite sets A, B, C respectively, with some (maybe empty) extra-information, and such that $\Phi(A) \subseteq \Psi(B) \cap \mathrm{X}(C)$.

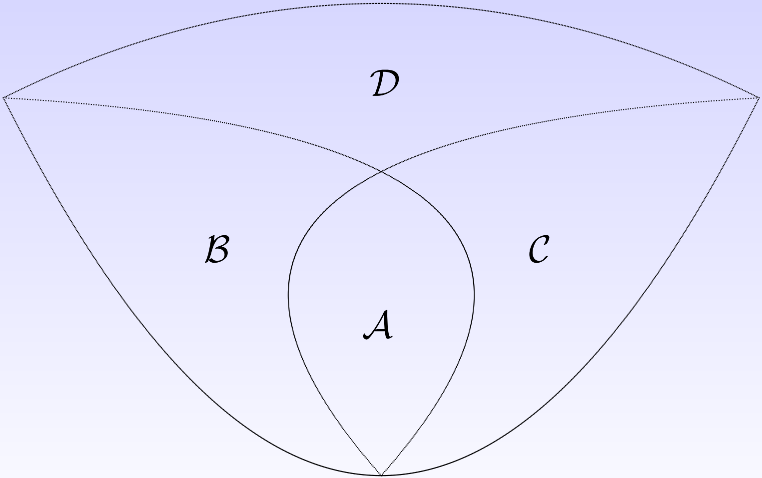
A (*syntactic*) *amalgam* of $\Psi(B)$ and $\mathrm{X}(C)$ over $\Phi(A)$ is a diagram $\Theta(D) \in \mathbf{T}_0$ such that $\Theta(D) \supseteq \Psi(B) \cup \mathrm{X}(C)$.

In particular, these diagrams can contain only inner descriptions for finite structures $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ with universes A, B, C. In such a case, a structure $\mathcal{D}$ with a universe D is a (*semantic*) amalgam of $\mathcal{B}$ and $\mathcal{C}$ over $\mathcal{A}$.

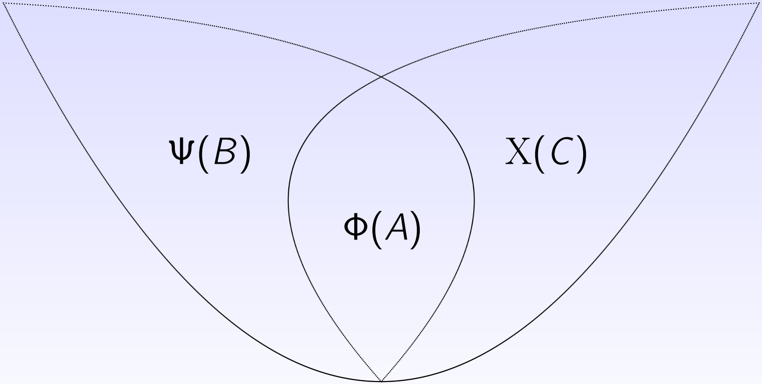
Semantic amalgam



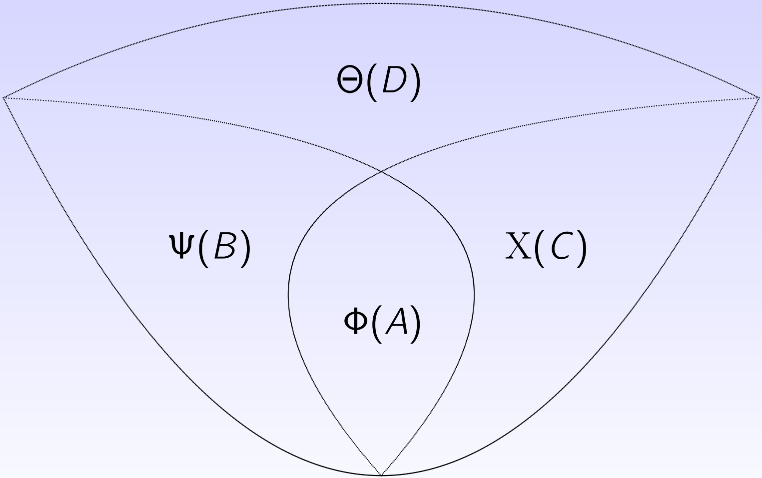
Semantic amalgam



Syntactic amalgam



Syntactic amalgam



We construct a *generic* structure $\mathcal{M}$ step-by-step using a given class $\mathbf{T}_0$ of diagrams and of their amalgams such that all diagrams in $\mathbf{T}_0$ are represented in $\mathcal{M}$:

(1) every finite set A_0 in the universe M of $\mathcal{M}$ is extensible to a finite set $A \subseteq M$ with a diagram $\Phi(A) \in \mathbf{T}_0$ satisfied in $\mathcal{M}$: $\mathcal{M} \models \Phi(A)$;

(2) if $A \subseteq M$ is a finite set, $\Phi(A), \Psi(B) \in \mathbf{T}_0$, $\mathcal{M} \models \Phi(A)$ and $\Phi(A) \leqslant \Psi(B)$ (where $\leqslant$ is a given upward directed partial order for $\mathbf{T}_0$ coordinated with $\subseteq$), then there exists a set $B' \subseteq M$ such that $A \subseteq B'$ and $\mathcal{M} \models \Psi(B')$.

Generic structures

Every finite part of the extra-information should be realized on some step: if $\exists x \varphi(x) \in \Phi(A)$ then there are $B \supseteq A$ with $\Psi(B) \supseteq \Phi(A)$ and an element $b \in B$ such that $\varphi(b) \in \Psi(B)$.

Finite steps approximate the required generic structure.

We form a required theory (with desirable properties) introducing an appropriate class $\mathbf{T}_0$ of (in)complete diagrams.

If the process is organized uniformly then diagrams $\Phi(A) \in \mathbf{T}_0$ force complete types $\mathrm{tp}(A)$.

It is natural to take diagrams with a minimal information including atomic links by predicates and operations.

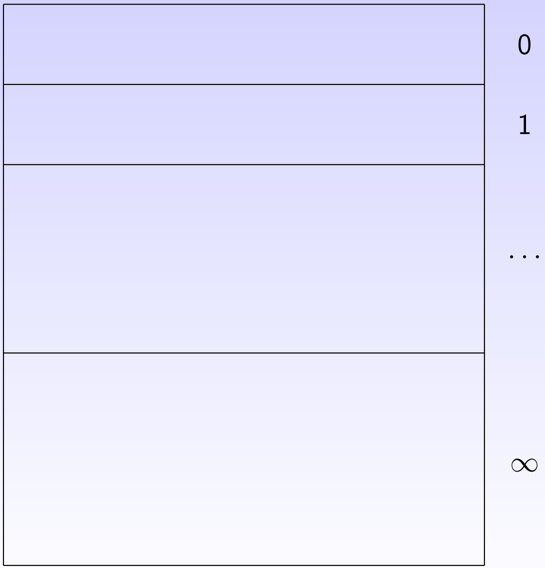
We put an information via the class $\mathbf{T}_0$ and obtain a *generic* theory T such that models of T realize the information I. Here, step-by-step we construct simultaneously a syntactic object T collecting the required information I and a semantic object $\mathcal{M}$ realizing I.

Generic constructions forcing given distributions of countable models

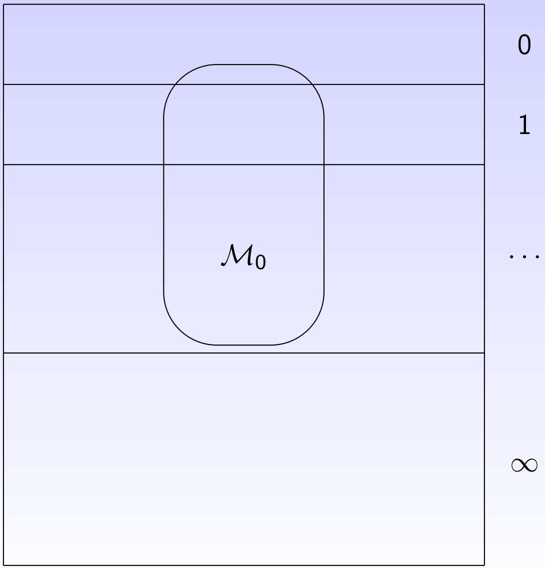
There are natural examples but they do not cover all characteristics.

We illustrate the mechanism for realizations of basic characteristics on the following examples.

$I(T,\omega)=3$, $P(T)=2$, $L(T)=1$



$I(T,\omega) = 3$, $P(T) = 2$, $L(T) = 1$



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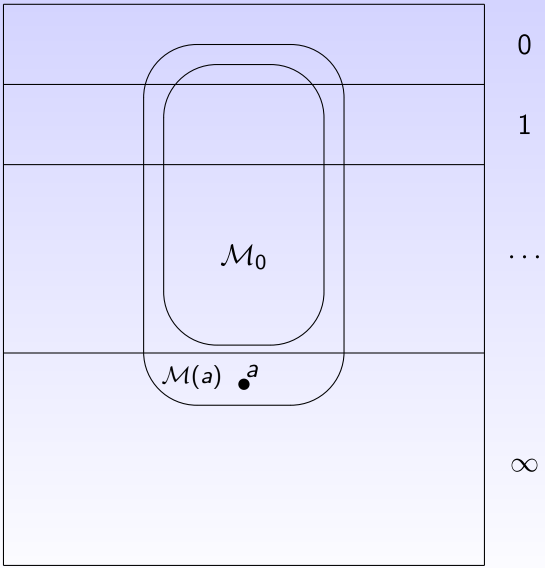

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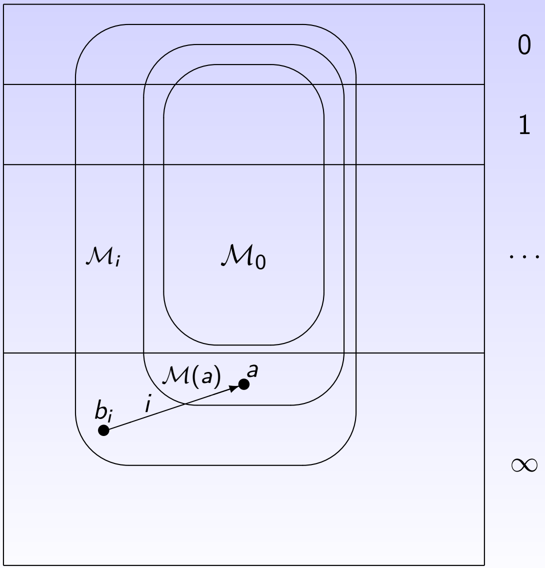
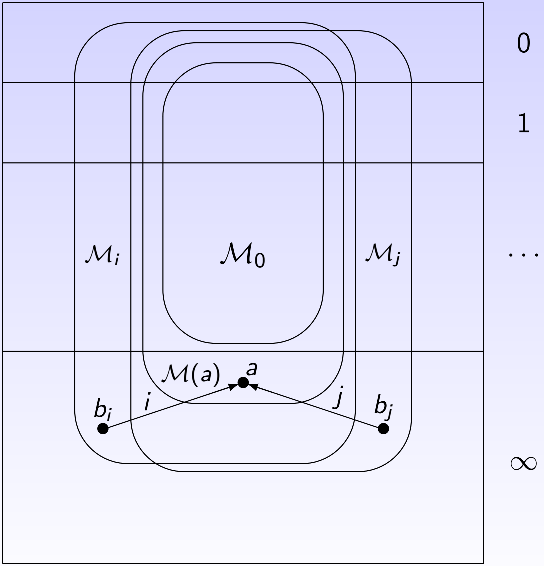
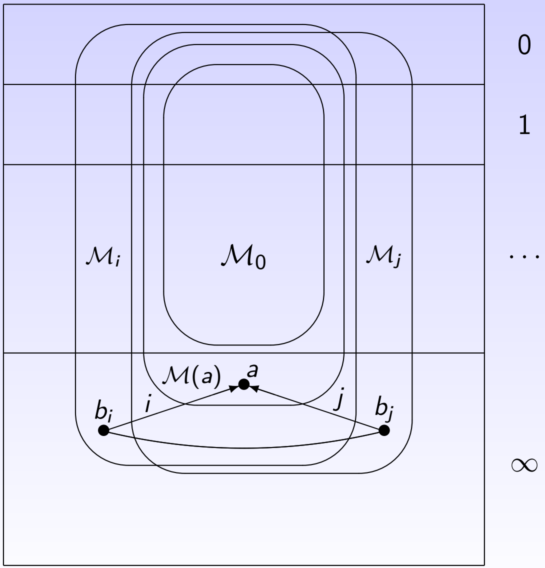


$I(T,\omega) = 3,\ P(T) = 2,\ L(T) = 1$



$I(T, \omega) = 3$, $P(T) = 2$, $L(T) = 1$



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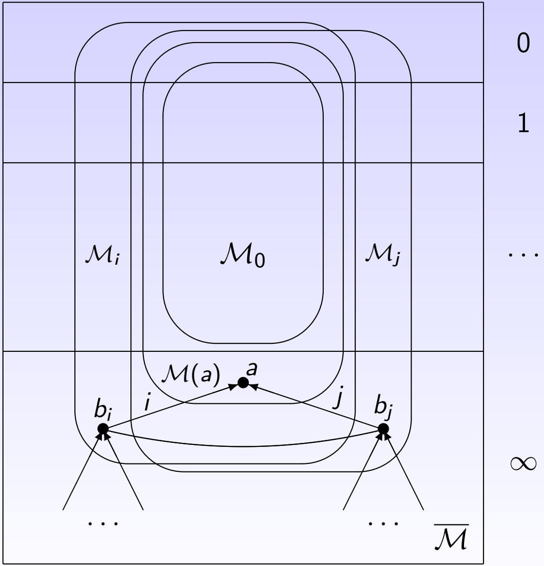


$I(T,\omega)=3$, $P(T)=2$, $L(T)=1$

We consider countably many disjoint colors $0, 1, \ldots, n, \ldots$ for elements and directed links by colored arcs also with countably many disjoint colors ordered by colors of elements.

For the theory T we have a prime model $\mathcal{M}_0$ whose all elements have finite colors.

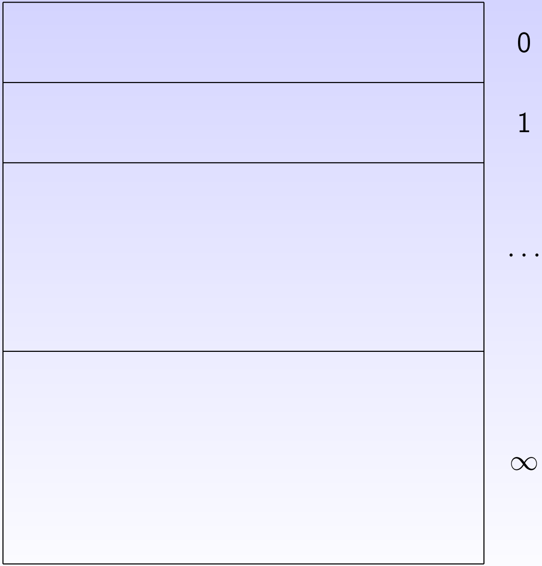
Furthermore, there is a prime model $\mathcal{M}(a)$ over a realization a of the type describing the infinite color.

The model $\mathcal{M}(a)$ has countably many essentially distinct extensions $\mathcal{M}(b_i)$ by arcs of colors i.

We introduce "bridges", i.e., principal edges $[b_i, b_j]$, guaranteing that any i-extension $\mathcal{M}_i = \mathcal{M}(b_i)$ is equal to a j-extension $\mathcal{M}_j = \mathcal{M}(b_j)$.

Having these bridges we obtain unique limit model $\overline{\mathcal{M}}$ (together with two non-isomorphic almost prime models $\mathcal{M}_0$ and $\mathcal{M}(a)$).

$I(T,\omega) = 4$, $P(T) = 2$, $L(T) = 2$



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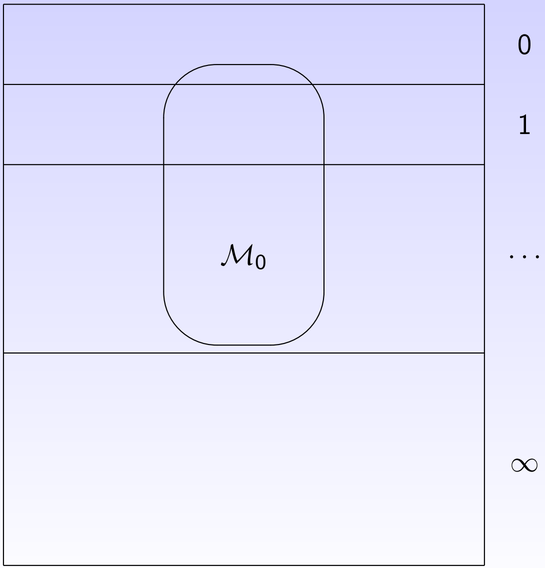

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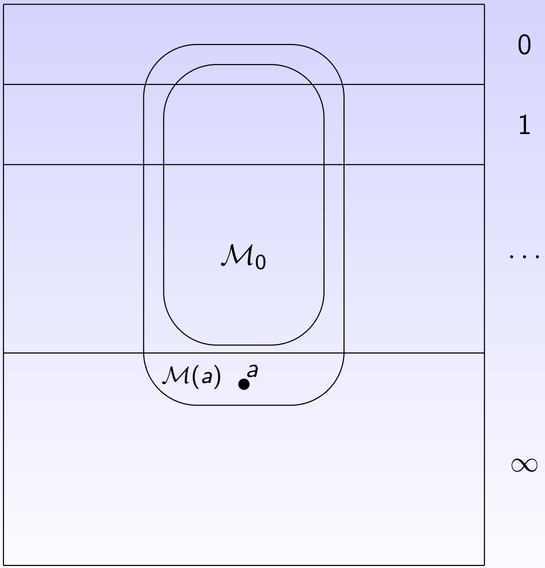

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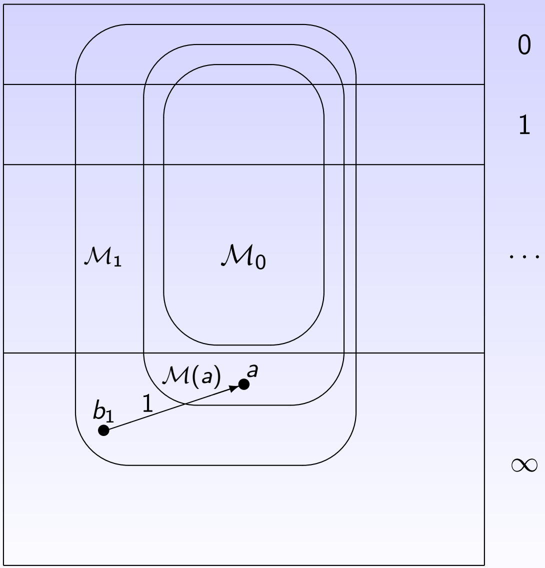

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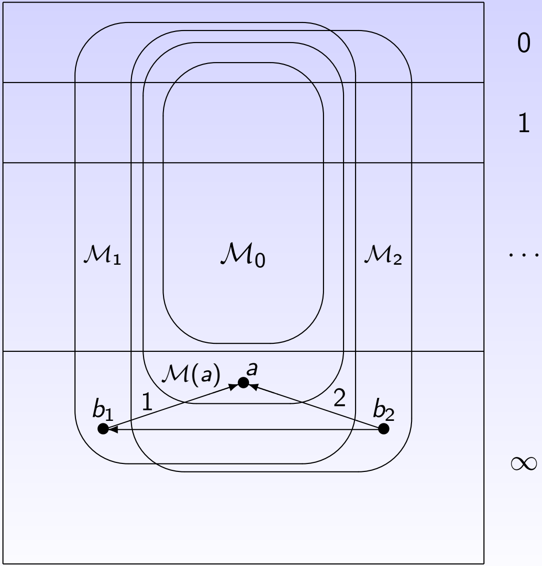

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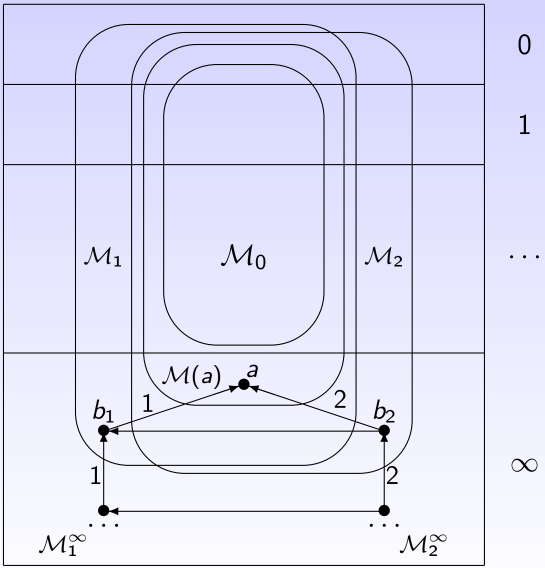


$I(T,\omega) = 4$, $P(T) = 2$, $L(T) = 2$

Basing on the example of theory with three countable models we consider again countably many disjoint colors $0, 1, \dots, n, \dots$ for elements and directed links by colored arcs also with countably many disjoint colors ordered by colors of elements.

For the theory T we have a prime model $\mathcal{M}_0$ whose all elements have finite colors.

Furthermore, there is a prime model $\mathcal{M}(a)$ over a realization a of the type describing the infinite color.

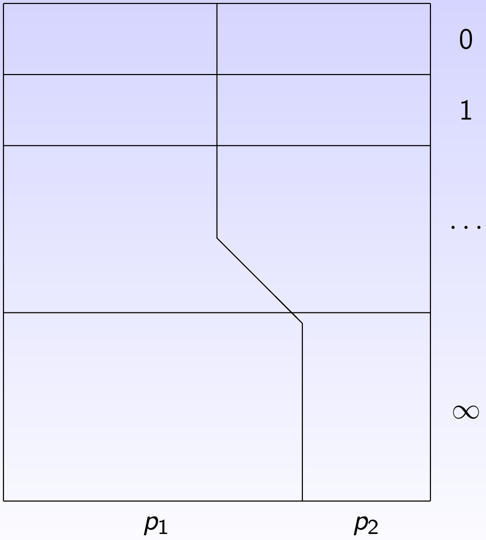
The model $\mathcal{M}(a)$ has two essentially distinct extensions $\mathcal{M}_1 = \mathcal{M}(b_1)$ and $\mathcal{M}_2 = \mathcal{M}(b_2)$ by arcs of colors 1 and 2 respectively.

$I(T,\omega)=4$, $P(T)=2$, $L(T)=2$

We introduce arcs (b_2, b_1) guaranteing that any 2-extension $\mathcal{M}_2$ includes a 1-extension $\mathcal{M}_1$ but not vice versa.

Having these arcs we obtain two limit models $\mathcal{M}_1^\infty$ and $\mathcal{M}_2^\infty$ corresponding to elementary chains of 1-extensions and of 2-extensions (together with two non-isomorphic almost prime models $\mathcal{M}_0$ and $\mathcal{M}(a)$). Here $\mathcal{M}_2^\infty$ is saturated.

$I(T,\omega) = 4$, $P(T) = 3$, $L(T) = 1$, linear $\leq_{\rm RK}$



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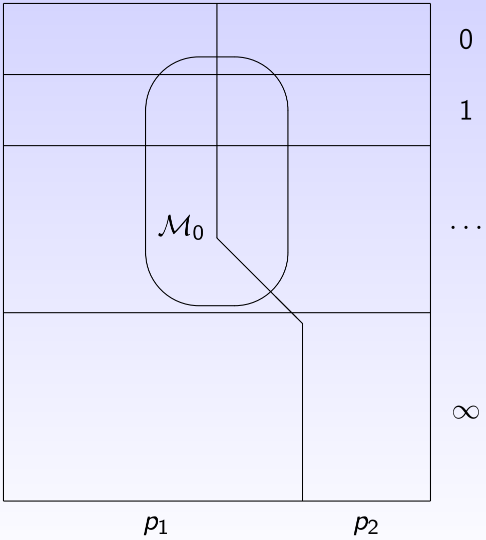

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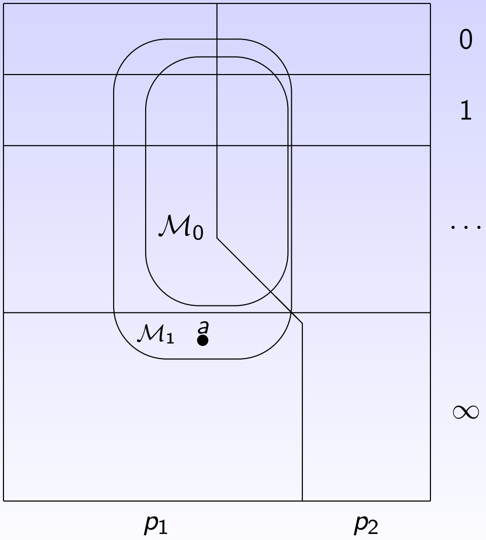

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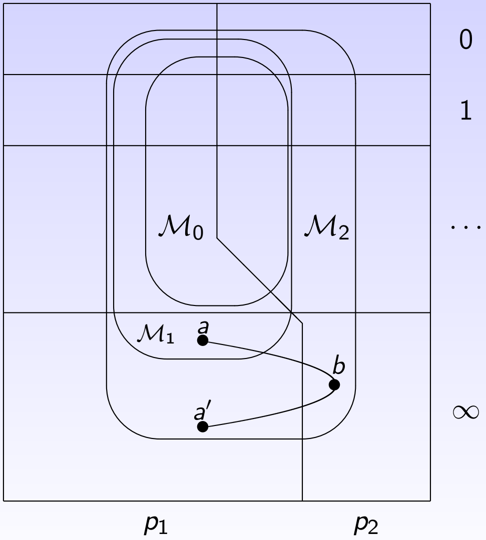

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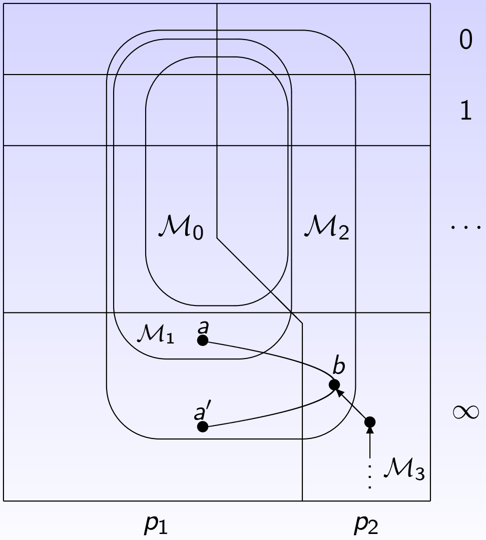


$I(T,\omega)=4$, $P(T)=3$, $L(T)=1$, linear $\leq_{\rm RK}$

Basing on the previous examples, for two disjoint infinite unary predicates P_0 and P_1, we consider countably many disjoint colors $0,1,\ldots,n,\ldots$ for elements and directed links by colored arcs also with countably many disjoint colors ordered by colors of elements. Thus we have two non-isolated 1-types p_1 and p_2 realizing predicates P_0 and P_1 by elements of infinite color.

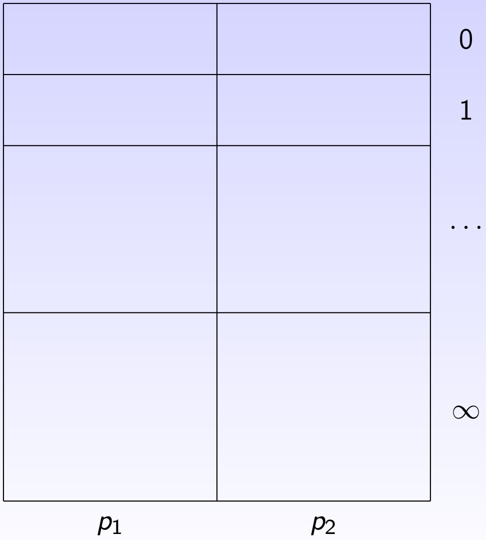
For the theory T we have a prime model $\mathcal{M}_0$ whose all elements have finite colors.

Furthermore, there is a prime model $\mathcal{M}_1=\mathcal{M}(a)$ over a realization a of the type p_1 such that $\mathcal{M}_1$ has a unique realization of p_1.

The model $\mathcal{M}(a)$ has an extension $\mathcal{M}_2=\mathcal{M}(b)$, where b is a realization of p_2. Moreover, having a and a realization $a'\neq a$ of p_1 we obtain a model isomorphic to $\mathcal{M}_2$, w.l.o.g. $\mathcal{M}(b)=\mathcal{M}(a,a')$.

Extending the model $\mathcal{M}_2$ by principal arcs we get unique limit model $\mathcal{M}_3$.

$I(T,\omega) = 4$, $P(T) = 3$, $L(T) = 1$, non-linear $\leq_{\mathrm{RK}}$



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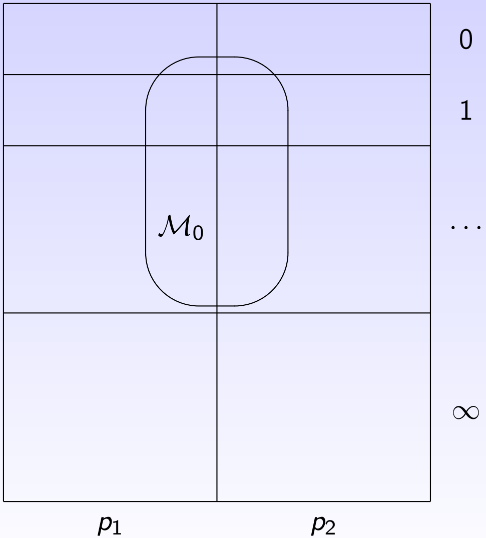

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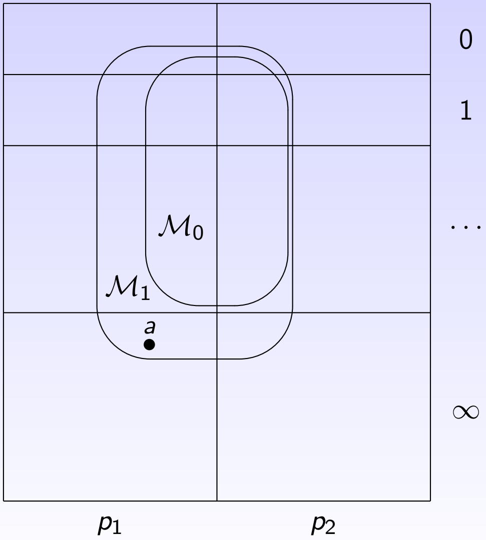

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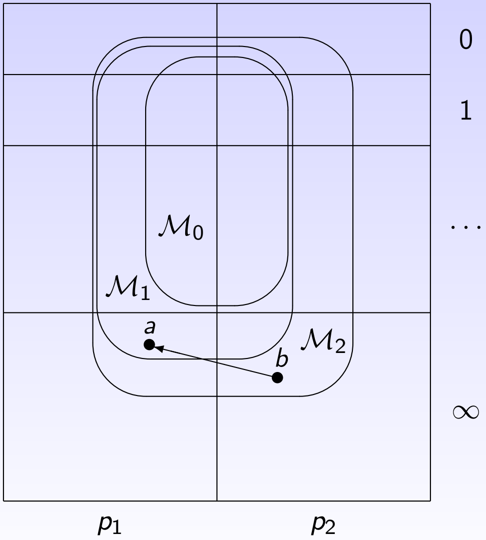

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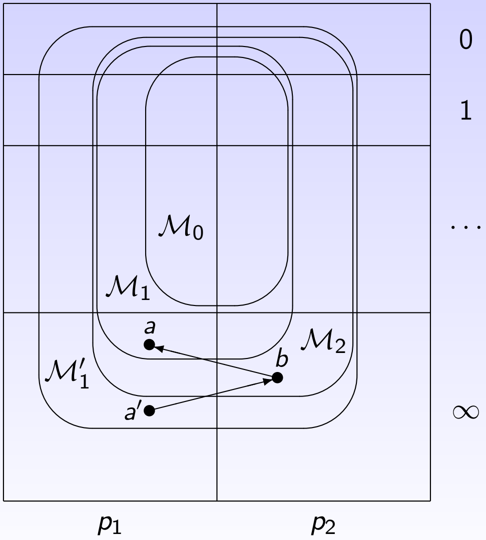
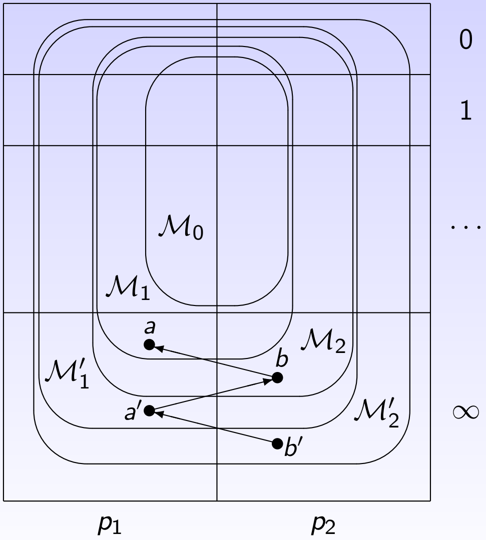
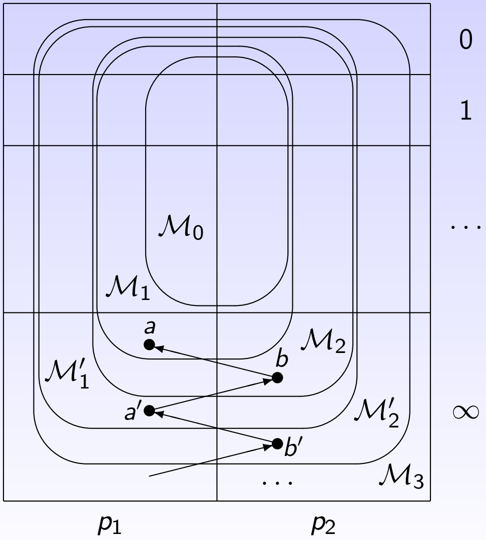


$I(T,\omega)=4$, $P(T)=3$, $L(T)=1$, non-linear $\leq_{\rm RK}$



$I(T,\omega) = 4$, $P(T) = 3$, $L(T) = 1$, non-linear $\leq_{\rm RK}$



$I(T, \omega) = 4$, $P(T) = 3$, $L(T) = 1$, non-linear $\leq_{\mathrm{RK}}$

Again for two disjoint infinite unary predicates P_0 and P_1, we consider countably many disjoint colors $0, 1, \ldots, n, \ldots$ for elements and directed links by colored arcs also with countably many disjoint colors ordered by colors of elements. We have two non-isolated 1-types p_1 and p_2 realizing predicates P_0 and P_1 by elements of infinite color.

For the theory T we have a prime model $\mathcal{M}_0$ whose all elements have finite colors.

Furthermore, there is a prime model $\mathcal{M}_1 = \mathcal{M}(a)$ over a realization a of the type p_1.

The model $\mathcal{M}(a)$ has a proper extension $\mathcal{M}_2 = \mathcal{M}(b)$ by a principal arc (b, a), where b is a realization of p_2 and $\mathcal{M}_2$ in not isomorphic to $\mathcal{M}_1$.

$I(T,\omega) = 4$, $P(T) = 3$, $L(T) = 1$, non-linear $\leq_{\rm RK}$

Then the model $\mathcal{M}_2$ has a proper extension $\mathcal{M}'_1 = \mathcal{M}(a')$ by a principal arc (a', b), where a' is a realization of p_1, $\mathcal{M}'_1$ has a proper extension $\mathcal{M}'_2 = \mathcal{M}(b')$ by a principal arc (b', a'), where b' is a realization of p_2, etc.

Extending the chain of almost prime models we get unique limit model $\mathcal{M}_3$.

Below we consider a series of derivative objects used for a classification of structures and their elementary theories:

- Rudin–Keisler preorders and distribution functions for limit models of a given theory, producing spectra of countable models (with B.S. Baizhanov, B.Sh. Kulpeshov, and V.V. Verbovskiy);
- Algebras for distributions of formulas over families of types (with B.Sh. Kulpeshov and D.Yu. Emel'yanov);
- Hypergraphs of models of a theory (with B.Sh. Kulpeshov);
- Generic classes and their limits (with P. Stefaneas, Y. Kiouvrekis, and A.Yu. Mikhaylenko);
- Topologies, closures, generating sets, e-spectra, and ranks for families of theories with respect to P-operators and E-operators (with In.I. Pavlyuk and N.D. Markhabatov).

Pseudo-supersimple theories [4] [5] [6]

Definition (A. Tsuboi)

A stable theory T is *pseudo-superstable* if T fails to have the infinite weight.

Similarly we say that a simple theory T is *pseudo-supersimple* if T fails to have the infinite weight.
We generalize both the Tsuboi theorem for pseudo-superstable theories and the Kim theorem for simple theories, obtaining

Theorem (B.S. Baizhanov, S.V. Sudoplatov, V.V. Verbovskiy, 2012)

Let T be a union of pseudo-supersimple theories T_n, where $T_n \subseteq T_{n+1}$, $n \in \omega$. Then $I(T, \omega) = 1$ or $I(T, \omega) \geq \omega$.

[4] A. Tsuboi, Countable models and unions of theories, J. Math. Soc. Japan. 1986. Vol. 38, No. 3. P. 501–508.

[5] B. Kim , On the number of countable models of a countable supersimple theory, J. London Math. Soc. 1999. Vol. 60, No. 2. P. 641–645.

[6] B. S. Baizhanov, S. V. Sudoplatov, V. V. Verbovskiy, Conditions for non-symmetric relations of semi-isolation, Siberian Electronic Mathematical Reports. 2012. Vol. 9. P. 161–184.

Weakly o-minimal structures[7]

Definition

A *weakly o-minimal structure* is a linearly ordered structure $\mathcal{M} = \langle M, =, <, \ldots \rangle$ such that any definable (with parameters) subset of the structure M is a finite union of convex sets in $\mathcal{M}$.

We recall that such a structure $\mathcal{M}$ is said to be *o-minimal* if any definable (with parameters) subset of $\mathcal{M}$ is the union of finitely many intervals and points in $\mathcal{M}$.

[7] H.D. Macpherson, D. Marker, and C. Steinhorn, Weakly o-minimal structures and real closed fields // Transactions of The American Mathematical Society, 352 (2000), pp. 5435–5483.

Weakly orthogonal types[8]

Definition

Assuming that $\mathcal{M}$ is an $|A|^+$-saturated weakly o-minimal structure, $A, B \subseteq M$, and $p, q \in S^1(A)$ are non-algebraic types, we say that p is not *weakly orthogonal* to q ($p \not\perp^w q$) if there are an A-definable formula $H(x, y)$, $a \in p(\mathcal{M})$, and $b_1, b_2 \in q(\mathcal{M})$ such that $b_1 \in H(\mathcal{M}, a)$ and $b_2 \notin H(\mathcal{M}, a)$.

Lemma (B.S. Baizhanov)

The relation $\not\perp^w$ of the weak non-orthogonality is an equivalence relation on $S^1(A)$.

[8] B.S. Baizhanov, Expansion of a model of a weakly o-minimal theory by a family of unary predicates // The Journal of Symbolic Logic, 66 (2001), pp. 1382–1414.

Quite o-minimal theories[9]

Definition

We say that p is not *quite orthogonal* to q ($p \not\perp^q q$) if there is an A-definable bijection $f : p(M) \to q(M)$.

We say that a weakly o-minimal theory is *quite o-minimal* if the relations of weak and quite orthogonality coincide for 1-types over arbitrary sets of models of the given theory.

[9] B.Sh. Kulpeshov, Convexity rank and orthogonality in weakly o-minimal theories // News of the National Academy of Sciences of the Republic of Kazakhstan, physical and mathematical series, 227 (2003), pp. 26–31.

Simple types[10]

Definition

A type $p \in S_1(\emptyset)$ is called *simple* if for any $n \in \omega$, any non-trivial $\emptyset$-definable n-ary function $f(x_1, \ldots, x_n)$, and any realizations $a_1, \ldots, a_n$ of p, $f(a_1, \ldots, a_n)$ does not realize the type p.

[10] L.L. Mayer, Vaught's conjecture for o-minimal theories // The Journal of Symbolic Logic, 53 (1988), pp. 146–159.

Convexity rank[11]

Definition

Let T be a weakly o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model of T, $\phi(x)$ be a M-definable formula with one free variable. The *convexity rank of the formula* $\phi(x)$ ($RC(\phi(x))$) is defined as follows:
1) $RC(\phi(x)) \geq 1$ if $\phi(M)$ is infinite;
2) $RC(\phi(x)) \geq \alpha + 1$ if there are a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence $b_i, i \in \omega$ of elements such that:

- for any $i, j \in \omega$, with $i \neq j$ we have $\mathcal{M} \models \neg E(b_i, b_j)$;
- for any $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(\mathcal{M}, b_i)$ is a convex subset of $\phi(\mathcal{M})$;

[11] B.Sh. Kulpeshov, Weakly o-minimal structures and some of their properties // The Journal of Symbolic Logic, 63 (1998), pp. 1511–1528.

Convexity rank

3) $RC(\phi(x)) \geq \delta$ if $RC(\phi(x)) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.
If $RC(\phi(x)) = \alpha$ for some α then we say that $RC(\phi(x))$ is defined. Otherwise (i.e., if $RC(\phi(x)) \geq \alpha$ for all α) we set $RC(\phi(x)) = \infty$.

The *convexity rank of* 1*-type* p ($RC(p)$) is the infimum of the set $\{RC(\phi(x)) \mid \phi(x) \in p\}$, i.e., $RC(p) := \inf\{RC(\phi(x)) \mid \phi(x) \in p\}$.

Binary convexity rank

The convexity rank of an arbitrary unary formula $\phi(x)$ is called *binary* and it is denoted by $RC_{bin}(\phi(x))$ if in Definition above the parametrically definable equivalence relations are replaced by $\emptyset$-definable (i.e., binary) equivalence relations.

Quite o-minimal theories with few countable models[12]

We say that a quite o-minimal theory T has *few countable models* if T has fewer than 2^ω countable models up to isomorphisms.

Theorem

Let T be a quite o-minimal theory in a countable language. Then either T has 2^ω countable models or T has exactly $3^k \cdot 6^s$ countable models, where k and s are natural numbers. Moreover, for any $k, s \in \omega$ there is a quite o-minimal theory T with exactly $3^k \cdot 6^s$ countable models.

Theorem above generalizes the known Mayer's theorem on countable models of o-minimal theories: L.L. Mayer, Vaught's conjecture for o-minimal theories // The Journal of Symbolic Logic, 53 (1988), pp. 146–159.

[12] B.Sh. Kulpeshov, S.V. Sudoplatov, Vaught's conjecture for quite o-minimal theories // Annals of Pure and Applied Logic, 2017. Vol. 168, N 1. P. 129-149.

Quite *o*-minimal theories with few countable models[13]

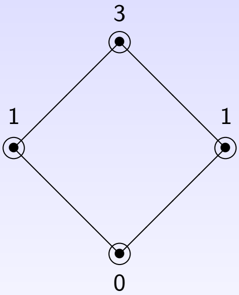
Theorem

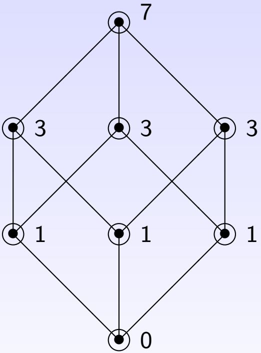
Let T be a quite o-minimal theory with few countable models. Then
(1) T is binary;
(2) each non-algebraic type $p \in S_1(\emptyset)$ is simple and has a finite convexity rank.

[13] B.Sh. Kulpeshov, S.V. Sudoplatov, Vaught's conjecture for quite o-minimal theories // Annals of Pure and Applied Logic, 2017. Vol. 168, N 1. P. 129-149.

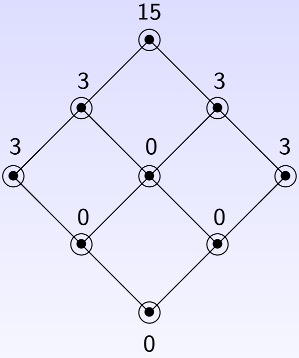
Quite o-minimal theories with $I(T,\omega)=3$ and $I(T,\omega)=6$

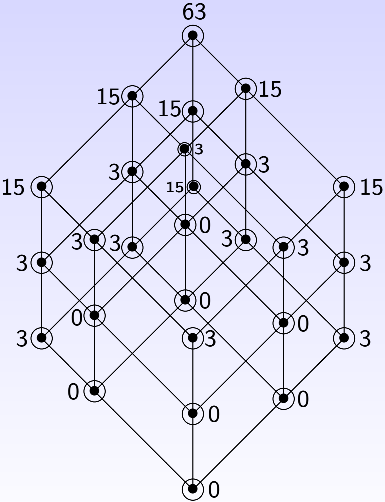
Quite o-minimal theories with $I(T,\omega) = 3^2$ and $I(T,\omega) = 3^3$



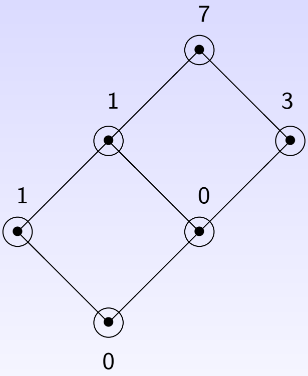


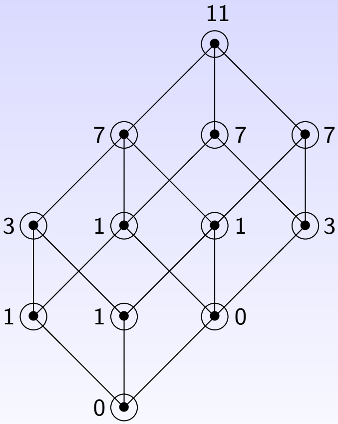
Quite o-minimal theories with $I(T,\omega) = 6^2$ and $I(T,\omega) = 6^3$



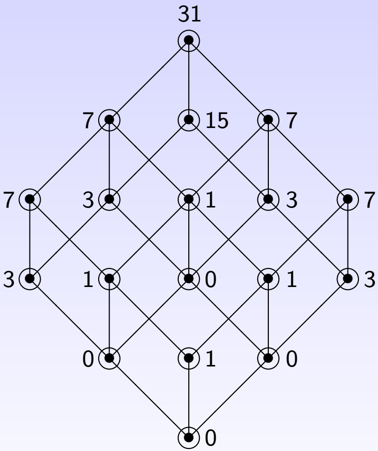


Quite o-minimal theories with $I(T,\omega)=3\cdot 6$ and $I(T,\omega)=3^2\cdot 6$





Quite o-minimal theories with $I(T,\omega) = 3 \cdot 6^2$



Decomposition formula (with B.Sh. Kulpeshov) [14] [15]

Applying Theorem on numbers of countable models for quite *o*-minimal theories we have the following representation of Decomposition formula (1):

$$3^k \cdot 6^s = 2^k \cdot 3^s + \sum_{t=0}^{k} \sum_{m=0}^{s} 2^{s-m} \cdot (2^t \cdot 4^m - 1) \cdot C_k^t \cdot C_s^m .$$

[14] B.Sh. Kulpeshov, S.V. Sudoplatov, Distributions of countable models of quite o-minimal Ehrenfeucht theories, arXiv:1802.08078v1 [math.LO]. 2018.

[15] S.V. Sudoplatov, On distributions of countable models of disjoint unions of Ehrenfeucht theories, Russian Mathematics. 2018. Vol. 62, No. 11. P.76-80.

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- S.V. Sudoplatov, Hypergraphs of prime models and distributions of countable models of small theories, J. Math. Sciences. 2010. Vol. 169, No. 5. P. 680–695.
- B.S. Baizhanov, S.V. Sudoplatov, V.V. Verbovskiy, Conditions for non-symmetric relations of semi-isolation, Siberian Electronic Mathematical Reports. 2012. Vol. 9. P. 161–184.

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- D.Yu. Emel'yanov, B.Sh. Kulpeshov, S.V. Sudoplatov, Algebras of binary formulas for compositions of theories. (submitted)

Definition. Let T be a complete theory, $\mathcal{M} \models T$. Consider types $p(x), q(y) \in S(\varnothing)$, realized in $\mathcal{M}$, and all (p, q)-*semi-isolating*, or (p, q)-*preserving* formulas $\varphi(x, y)$ of T, i. e., formulas for which there is $a \in M$ such that $\models p(a)$ and $\varphi(a, y) \vdash q(y)$. Now, for each such a formula $\varphi(x, y)$, we define a binary relation $R_{p,\varphi,q} \rightleftharpoons \{(a, b) \mid \mathcal{M} \models p(a) \wedge \varphi(a, b)\}$. If $(a, b) \in R_{p,\varphi,q}$, then (a, b) is called a (p, φ, q)-*arc*. If $\varphi(a, y)$ is principal (over a), the (p, φ, q)-arc (a, b) is also *principal*. If besides $\varphi(x, b)$ is a principal formula (over b) then the set $[a, b] \rightleftharpoons \{(a, b), (b, a)\}$ is a *principal* (p, φ, q)-edge.

Definitions

For types $p(x), q(y) \in S(\varnothing)$, we denote by $\mathrm{PF}(p, q)$ the set

$$\{\varphi(x, y) \mid \varphi(a, y) \text{ is a principal formula, } \varphi(a, y) \vdash q(y), \text{ where } \models p(a)\}.$$

Let $\mathrm{PE}(p, q)$ be the set of all pairs $(\varphi(x, y), \psi(x, y))$ of formulas in $\mathrm{PF}(p, q)$ such that for any (some) realization a of p the sets of solutions for $\varphi(a, y)$ and $\psi(a, y)$ coincide.
Clearly, $\mathrm{PE}(p, q)$ is an equivalence relation on the set $\mathrm{PF}(p, q)$.

Definitions

Thus the quotient $\mathrm{PF}(p,q)/\mathrm{PE}(p,q)$ is represented as a disjoint union of sets $\mathrm{PFS}(p,q)$ and $\mathrm{PFN}(p,q)$, where $\mathrm{PFS}(p,q)$ consists of $\mathrm{PE}(p,q)$-classes corresponding to principal edges and $\mathrm{PFN}(p,q)$ consists of $\mathrm{PE}(p,q)$-classes corresponding to irreversible principal arcs.

Definitions

Let T be a complete theory, $U = U^- \dot\cup \{0\} \dot\cup U^+$ be an alphabet of cardinality $\geq |S(T)|$, consisting of *negative elements* $u^- \in U^-$, *positive elements* $u^+ \in U^+$, and zero 0. As usual, we write $u < 0$ for any $u \in U^-$ and $u > 0$ for any $u \in U^+$. The set $U^- \cup \{0\}$ is denoted by $U^{\leq 0}$ and $U^+ \cup \{0\}$ is denoted by $U^{\geq 0}$. Elements of U are called *labels*.

Definitions

Let $\nu(p,q)\colon \mathrm{PF}(p,q)/\mathrm{PE}(p,q) \to U$ be an injective *labelling functions*, $p(x)$, $q(y) \in S(\varnothing)$, for which negative elements correspond to classes in $\mathrm{PFN}(p,q)/\mathrm{PE}(p,q)$ and non-negative elements correspond to classes in $\mathrm{PFS}(p,q)/\mathrm{PE}(p,q)$ such that 0 is defined only for $p = q$ and is represented by the formula $(x \approx y)$, $\nu(p) \rightleftharpoons \nu(p,p)$. We additionally suppose that $\rho_{\nu(p)} \cap \rho_{\nu(q)} = \{0\}$ for $p \neq q$ (where, as usual, we denote by ρ_f the image of the function f) and $\rho_{\nu(p,q)} \cap \rho_{\nu(p',q')} = \varnothing$ if $p \neq q$ and $(p,q) \neq (p',q')$. Labelling functions with the properties above as well families of these functions are said to be *regular*. Below we shall consider only regular labelling functions and their regular families.

Definitions

We denote by $\theta_{p,u,q}(x,y)$ a formula in $\mathrm{PF}(p,q)$ with the label $u\in\rho_{\nu(p,q)}$. If a type p is fixed and $p=q$ then a formula $\theta_{p,u,q}(x,y)$ is denoted by $\theta_u(x,y)$.
For types $p_1,p_2,\ldots,p_{k+1}\in S^1(\varnothing)$ and sets $X_1,X_2,\ldots,X_k\subseteq U$ of labels we denote by

$$P(p_1,X_1,p_2,X_2,\ldots,p_k,X_k,p_{k+1})$$

the set of all labels $u\in U$ corresponding to formulas $\theta_{p_1,u,p_{k+1}}(x,y)$ satisfying, for realizations a of p_1 and some $u_1\in X_1,\ldots,u_k\in X_k$, the following condition:

$$\theta_{p_1,u,p_{k+1}}(a,y)\vdash\theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(a,y),$$

где

$$\theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(x,y)\rightleftharpoons$$
$$\rightleftharpoons\exists x_2,x_3,\ldots x_k(\theta_{p_1,u_1,p_2}(x,x_2)\wedge\theta_{p_2,u_2,p_3}(x_2,x_3)\wedge\ldots$$
$$\ldots\wedge\theta_{p_{k-1},u_{k-1},p_k}(x_{k-1},x_k)\wedge\theta_{p_k,u_k,p_{k+1}}(x_k,y)).$$

Definitions

Thus the Boolean $\mathcal{P}(U)$ of U is the universe of an *algebra of distributions of binary isolating formulas* with k-ary operations

$$P(p_1, \cdot, p_2, \cdot, \ldots, p_k, \cdot, p_{k+1}),$$

where $p_1, \ldots, p_{k+1} \in S^1(\varnothing)$.

Definitions

If all types p_i equal to a type p then we write $P_p(X_1, X_2, \ldots, X_k)$ and $P_p(u_1, u_2, \ldots, u_k)$ as well as $\lfloor X_1, X_2, \ldots, X_k \rfloor_p$ and $\lfloor u_1, u_2, \ldots, u_k \rfloor_p$ instead of

$$P(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1})$$

and

$$P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1})$$

respectively.
We set $\mathfrak{P}_{\nu(p)} \rightleftharpoons \langle \mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}; \lfloor \cdot, \cdot \rfloor_p \rangle$. The groupoid $\mathfrak{P}_{\nu(p)}$ is called the *groupoid of binary isolating formulas over the labelling function* $\nu(p)$ or the $I_{\nu(p)}$*-groupoid*. Below the operation $\lfloor \cdot, \cdot \rfloor$ will be also denoted by $\cdot$ and we write uv instead of $u \cdot v$.

Definitions

Since by the choice of the label 0 for the formula $(x \approx y)$ the equalities $X \cdot \{0\} = X$ and $\{0\} \cdot X = X$ are true for any $X \subseteq \rho_{\nu(p)}$, the groupoid $\mathfrak{P}_{\nu(p)}$ has the unit $\{0\}$, and it is a monoid if the algebra is right semi-associative. We have

$$Y \cdot Z = \bigcup \{yz \mid y \in Y, z \in Z\}$$

for any sets $Y, Z \in \mathcal{P}(\rho_{\nu(p)}) \setminus \{\varnothing\}$ in this structure.

Groupoids $\mathfrak{P}_{\nu(p)}$ are naturally extensible to groupoids $\mathfrak{P}_{\nu(R)}$ for binary isolating formulas on nonempty families $R \subseteq S^1(\emptyset)$ of types.

Quasirational and irrational types[16]

Definition

Let M be a weakly o-minimal structure, $A \subseteq M$, $p \in S_1(A)$ be a non-algebraic type.
We say that p is *quasirational to the right (left)* is there is an A-definable convex formula $U_p(x) \in p$ such that $U_p(N)^+ = p(N)^+$ ($U_p(N)^- = p(N)^-$) for a sufficiently saturated model $N \succ M$.
A non-isolated 1-type ia called *quasirational* if it is either quasirational to the right or quasirational to the left.
A non-quasirational non-isolated 1-type is called *irrational*.

[16] B.S. Baizhanov, Expansion of a model of a weakly o-minimal theory by a family of unary predicates // The Journal of Symbolic Logic, 66 (2001), pp. 1382–1414.

Monoids of isolating formulas

We construct by induction a sequence of embedded monoids $\mathfrak{A}_n$, $n \in \omega$.

For $\mathfrak{A}_0$, we take the algebra of distributions of isolating formulas with unique label 0 and the operation defined by equation $0 \cdot 0 = \{0\}$ (the algebra for an algebraic type r).

The algebra $\mathfrak{A}_1$ (for $RC(r) = 1$) is defined by the following table:

$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{1\}$	$\{0,1,2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{2\}$

Monoids of isolating formulas

For $\mathfrak{A}_2$ (with $RC(r) = 2$), we take the monoid defined by the following table:

$\cdot$	0	1	2	3	4
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
1	$\{1\}$	$\{1\}$	$\{0,1,2\}$	$\{3\}$	$\{4\}$
2	$\{2\}$	$\{0,1,2\}$	$\{2\}$	$\{3\}$	$\{4\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,1,2,3,4\}$
4	$\{4\}$	$\{4\}$	$\{4\}$	$\{0,1,2,3,4\}$	$\{4\}$

The monoids $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ form the cain $\mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \mathfrak{A}_2$.

Monoids of isolating formulas

If the monoid $\mathfrak{A}_n$ (for $RC(r) = n$) is already constructed, we define its extension $\mathfrak{A}_{n+1}$ (for $RC(r) = n + 1$) by adding of the labels $2n + 1$ and $2n + 2$, with the operation $\cdot$ on Boolean of the set $\{0, 1, \ldots, 2n + 2\}$ defined by the following table:

$\cdot$	0	1	2	...	$2n+1$	$2n+2$
0	$\{0\}$	$\{1\}$	$\{2\}$	...	$\{2n+1\}$	$\{2n+2\}$
1	$\{1\}$	$\{1\}$	$\{0,1,2\}$	...	$\{2n+1\}$	$\{2n+2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{2\}$	...	$\{2n+1\}$	$\{2n+2\}$
...	...	...	...	...	...	...
$2n+1$	$\{2n+1\}$	$\{2n+1\}$	$\{2n+1\}$	...	$\{2n+1\}$	$\{0,1,\ldots,2n+2\}$
$2n+2$	$\{2n+2\}$	$\{2n+2\}$	$\{2n+2\}$	...	$\{0,1,\ldots,2n+2\}$	$\{2n+2\}$

Monoids for countably categorical weakly o-minimal theories[17]

Any monoid, isomorphic to a monoid $\mathfrak{A}_n$, is called the *monoid of isolating formulas for a 1-type of a countably categorical weakly o-minimal theory, having the binary convexity rank n*, or briefly the $(P, \aleph_0, n)$-*wom-monoid*.

Theorem

Let T be a countably categorical weakly o-minimal theory. Then for any type $r \in S^1(\emptyset)$ and a natural n the following conditions are equivalent:
(1) the algebra $\mathfrak{P}_{\nu(r)}$ is a $(P, \aleph_0, n)$-wom-monoid;
(2) $RC_{bin}(r) = n$.

[17] D.Yu. Emel'yanov, B.Sh. Kulpeshov, S.V. Sudoplatov, Algebras of distributions for binary formulas in countably categorical weakly o-minimal structures // Algebra and Logic. 2017. V. 56, N 1. P. 13-36.

The monoid of binary isolating formulas for a quasirational to the right 1-type p, with the convexity rank n, in an arbitrary weakly o-minimal theory, is called the (P, QR, n)-*wom-monoid* and denoted by $\mathfrak{A}_n^{QR}$.

Similarly, the monoid of binary isolating formulas for a quasirational to the left 1-type, with the convexity rank n, is called the (P, QL, n)-*wom-monoid* and denoted by $\mathfrak{A}_n^{QL}$.

We assert that the type p has the algebra $\mathfrak{A}_n^{QR}$ of isolating formulas, which consists of $2n$ labels and defined by the following table:

$\cdot$	0	1	2	$\ldots$	$2n-3$	$2n-2$	-1
0	$\{0\}$	$\{1\}$	$\{2\}$	$\ldots$	$\{2n-3\}$	$\{2n-2\}$	$\{-1\}$
1	$\{1\}$	$\{1\}$	$\{0,1,2\}$	$\ldots$	$\{2n-3\}$	$\{2n-2\}$	$\{-1\}$
2	$\{2\}$	$\{0,1,2\}$	$\{2\}$	$\ldots$	$\{2n-3\}$	$\{2n-2\}$	$\{-1\}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\{-1\}$
$2n-3$	$\{2n-3\}$	$\{2n-3\}$	$\{2n-3\}$	$\ldots$	$\{2n-3\}$	$\{0,1,\ldots,2n-2\}$	$\{-1\}$
$2n-2$	$\{2n-2\}$	$\{2n-2\}$	$\{2n-2\}$	$\ldots$	$\{0,1,\ldots,2n-2\}$	$\{2n-2\}$	$\{-1\}$
-1	$\{-1\}$	$\{-1\}$	$\{-1\}$	$\ldots$	$\{-1\}$	$\{-1\}$	$\{-1\}$

Replacing the sign $<$, in the structure $\mathcal{M}$ and in the formulas θ, by $>$, we obtain the algebra $\mathfrak{A}_n^{QL}$ with the same table.
Notice that the algebras $\mathfrak{A}_n^{QR}$ and $\mathfrak{A}_n^{QL}$ are really the monoids. Here the monoids $\mathfrak{A}_n^{QR}$ and $\mathfrak{A}_n^{QL}$ have submonoids generated by non-negative labels $0, 1, \ldots, 2n-2$ and by non-positive labels $0, -1$.

We assert that an irrational type p of the convexity rank n has the algebra $\mathfrak{A}^I_n$ of isolating formulas, which consists of $2n-1$ labels and defined by the following table:

$\cdot$	0	1	2	$\dots$	$2n-3$	$2n-2$
0	$\{0\}$	$\{1\}$	$\{2\}$	$\dots$	$\{2n-3\}$	$\{2n-2\}$
1	$\{1\}$	$\{1\}$	$\{0,1,2\}$	$\dots$	$\{2n-3\}$	$\{2n-2\}$
2	$\{2\}$	$\{0,1,2\}$	$\{2\}$		$\dots$	$\{2n-3\}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$2n-3$	$\{2n-3\}$	$\{2n-3\}$	$\{2n-3\}$	$\dots$	$\{2n-3\}$	$\{0,1,\dots,2n-2\}$
$2n-2$	$\{2n-2\}$	$\{2n-2\}$	$\{2n-2\}$	$\dots$	$\{0,1,\dots,2n-2\}$	$\{2n-2\}$

Clearly, the algebra $\mathfrak{A}^I_n$ is isomorphic to the $(P,\aleph_0,n)$-wom-monoid $\mathfrak{A}_{n-1}$.

Monoids for quite o-minimal theories with few countable models[18]

Theorem (D.Yu. Emel'yanov, B.Sh. Kulpeshov, S.V. Sudoplatov)

Let T be a quite o-minimal theory with few countable models, $p \in S_1(\emptyset)$. Then there is $n < \omega$ such that:
(1) if p is isolated then the algebra $\mathfrak{P}_{\nu(p)}$ is the $(P, \aleph_0, n)$-wom-monoid consisting of $2n + 1$ labels;
(2) is p is quasirational to the right (left) then the algebra $\mathfrak{P}_{\nu(p)}$ is the (P, QR, n)-wom-monoid ((P, QL, n)-wom-monoid) consisting of $2n$ labels;
(3) if p is irrational then the algebra $\mathfrak{P}_{\nu(p)}$ is the (P, I, n)-wom-monoid consisting of $2n - 1$ labels.

[18] D.Yu. Emel'yanov, B.Sh. Kulpeshov, S.V. Sudoplatov, On algebras of distributions of binary formulas for quite o-minimal theories, Algebra and Logic. 2018. Vol. 57, No. 6. P. 429-444.

Corollary (D.Yu. Emel'yanov, B.Sh. Kulpeshov, S.V. Sudoplatov)

Let T be a quite o-minimal theory with few countable models, $p, q \in S_1(\emptyset)$. Then the algebras $\mathfrak{P}_{\nu(p)}$ and $\mathfrak{P}_{\nu(q)}$ are isomorphic if and only if $RC(p) = RC(q)$ and the types p and q are simultaneously either isolated, or quasirational, or irrational.

Algebras of formulas for families of 1-types[19]

Definition

We say that the algebra $\mathfrak{P}_{\nu(\{p,q\})}$ is *generalized commutative* if there is a bijection $\pi : \rho_{\nu(p)} \to \rho_{\nu(q)}$ witnessing that the algebras $\mathfrak{P}_{\nu(p)}$ and $\mathfrak{P}_{\nu(q)}$ are isomorphic (i.e., the defining tables coincide up to π) and for any $l \in \rho_{\nu(p,q)}$, $m \in \rho_{\nu(q,p)}$ we have $\pi(l \cdot m) = m \cdot l$.

[19] D.Yu. Emel'yanov, B.Sh. Kulpeshov, S.V. Sudoplatov, Algebras of distributions for binary formulas in countably categorical weakly o-minimal structures // Algebra and Logic. 2017. V. 56, N 1. P. 13-36.

Algebras of formulas for families of 1-types of countably categorical weakly o-minimal theories[20]

Theorem (D.Yu. Emel'yanov, B.Sh. Kulpeshov, S.V. Sudoplatov)

Let T be a countably categorical weakly o-minimal theory, $p, q \in S^1(\emptyset)$. Then the following conditions are equivalent:
(1) the algebra $\mathfrak{P}_{\nu(\{p,q\})}$ is a generalized commutative monoid;
(2) $RC_{bin}(p) = RC_{bin}(q)$.

[20] D.Yu. Emel'yanov, B.Sh. Kulpeshov, S.V. Sudoplatov, Algebras of distributions for binary formulas in countably categorical weakly o-minimal structures // Algebra and Logic. 2017. V. 56, N 1. P. 13-36.

Algebras of formulas for families of 1-types of quite o-minimal theories with few countable models

Theorem (D.Yu. Emel'yanov, B.Sh. Kulpeshov, S.V. Sudoplatov)

Let T be a quite o-minimal theory with few countable models, $p, q \in S^1(\emptyset)$, $p \not\perp^w q$. Then the algebra $\mathfrak{P}_{\nu(\{p,q\})}$ is a generalized commutative monoid.

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P-combinations

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the *P-union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the *P-operator*. The structure $\mathcal{A}_P$ is called the *P-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P|_{A_i})|_{\Sigma(\mathcal{A}_i)}$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

P-combinations

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}'|_{P_i})|_{\Sigma(\mathcal{A}_i)}$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}'|_{\bigcap\limits_{i\in I} \overline{P_i}}$, maybe applying Morleyzation. Moreover, we write $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$ for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

P-combinations

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_P = \mathrm{Th}(\mathcal{A}_P)$, being *$P$-combination* of T_i, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$. In general, for non-disjoint case, the theory T_P will be also called a *P-combination* of the theories T_i, but in such a case we will keep in mind that this P-combination is constructed with respect (and depending) to the structure $\mathcal{A}_P$, or, equivalently, with respect to any/some $\mathcal{A}' \equiv \mathcal{A}_P$.

E-combinations

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the *E-union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the *E-operator*. The structure $\mathcal{A}_E$ is also called the *E-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E|_{A_i})|_{\Sigma(\mathcal{A}_i)}$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes. The E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, being *$E$-combination* of T_i, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$ or by $\mathrm{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the *E-representability*.

E-combinations

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

e-spectrum

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by $e\text{-}\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}') \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by $e\text{-}\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$.

e-prime, e-minimal, and e-least structures

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

Definition. Let $\overline{\mathcal{T}}$ be the class of all complete elementary theories of relational languages. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i \in I}$, $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T} \subset \overline{\mathcal{T}}$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subseteq \mathcal{T}$.

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$.

Proposition

If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an *accumulation point* for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any formula $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.

Least generating sets

Theorem

If $\mathcal{T}_0'$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:
(1) $\mathcal{T}_0'$ is the least generating set for $\mathcal{T}_0$;
(2) $\mathcal{T}_0'$ is a minimal generating set for $\mathcal{T}_0$;
(3) any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0')_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0')_\varphi = \{T\}$;
(4) any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.

Ranks for families of theories

For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, for finite nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) = 0$, and for infinite families $\mathcal{T}$ — $\mathrm{RS}(\mathcal{T}) \geq 1$.
For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.
If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.
We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.
If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

Totally transcendental families of theories

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.
If $\mathcal{T}$ is totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Theorem

For any family $\mathcal{T}$ with $|\Sigma(\mathcal{T})| \leq \omega$ the following conditions are equivalent:
(1) $|\mathrm{Cl}_E(\mathcal{T})| = 2^\omega$;
(2) e-$\mathrm{Sp}(\mathcal{T}) = 2^\omega$;
(3) $\mathrm{RS}(\mathcal{T}) = \infty$.

Families of all theories of given languages

Theorem (N.D. Markhabatov, S.V. Sudoplatov, 2019)

For any language Σ either $\mathrm{RS}(\mathcal{T}_\Sigma)$ is finite, if Σ consists of finitely many 0-ary and unary predicates, and finitely many constant symbols, or $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$, otherwise.

Ranks and algebras for families of theories

Theorem (N.D. Markhabatov, S.V. Sudoplatov, 2019)

For any nonempty E-closed family $\mathcal{T}$, an ordinal $\alpha \geq 1$, and $n \in \omega \setminus \{0\}$, the following conditions are equivalent:
(1) $\mathrm{RS}(\mathcal{T}) = \alpha$ and $\mathrm{ds}(\mathcal{T}) = n$;
(2) the Boolean algebra $\mathcal{B}_s(\mathcal{T})$ is isomorphic to a direct product of n Boolean algebras $\mathcal{B}_1, \ldots, \mathcal{B}_n$ each of which is generated by elements of ranks $< \alpha$ such that each $\mathcal{B}_i$ contains infinitely many elements of each rank $\beta < \alpha$;
(3) the algebra $\mathcal{B}_d(\mathcal{T})$ consists of infinitely many atomic elements of each rank $\beta < \alpha$, $\beta \geq 0$, and exactly n atomic elements of rank α, these n atomic elements correspond non-principal ultrafilters with respect to elements of ranks $< \alpha$.